# Conecting Finsler Geometry and Mechanics via Geodesics

Guilherme Cerqueira Gonçalves - IME-USP
(FAPESP grants 2021/00551-3 and 2019/22488-1)
Advisor: Prof. Marcos M. Alexandrino - IME-USP

February 2022

### Definition 1 (Manifold)

A subset $M \subset \mathbb{R}^{m+k}$ is a **embedded manifold** if it is locally described by local graphs.

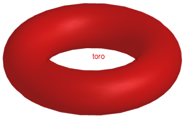
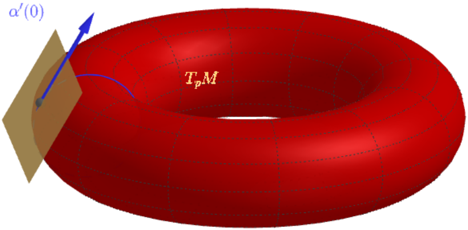



Figura: $M = \{x \in \mathbb{R}^3 | \left(\sqrt{x_1^2 + x_2^2} - 3\right)^2 + x_3^2 = 1\}$

## Definition 2 (Tangent Space)

The **Tangent Space** $T_pM$ is the vector space of **velocities** $\alpha'(0)$ of curves $\alpha : (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$. $TM = \cup_{p \in M} T_pM$ is the **Tangent Bundle** of $M$



## Definition 3 (Induced Riemannian Metric)

For each $p \in M$, and $V = (v_1, v_2, v_3), W = (w_1, w_2, w_3) \in T_pM$ (vectors tangent to $M$ at $p \in M$) we can define an inner product (called **metric at p**) at $T_pM$

$$g_p(V, W) = \langle V, W \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3$$

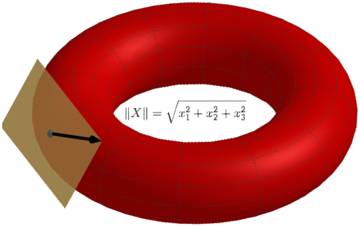



Figura: $\|X\| = \sqrt{g_p(X, X)}$

## Definition 4 (Randers Norm)

$R(v) = \sqrt{g(v,v)} + g(v,\vec{\beta})$ such that $\mathrm{g}$ is a Riemannian metric, and $\sqrt{g(\vec{\beta},\vec{\beta})} < 1$.

Definition 4 (Randers Norm)

$R(v) = \sqrt{g(v,v)} + g(v,\vec{\beta})$ such that $\mathrm{g}$ is a Riemannian metric, and $\sqrt{g(\vec{\beta},\vec{\beta})} < 1$.

Note that: (for $\lambda > 0$)

$$R(\lambda v) = \sqrt{g(\lambda v, \lambda v)} + g(\lambda v, \vec{\beta}) = \lambda(\sqrt{g(v,v)}) + \lambda(g(v,\vec{\beta})) = \lambda R(v)$$

### Definition 4 (Randers Norm)

$R(v) = \sqrt{g(v,v)} + g(v,\vec{\beta})$ such that $\mathrm{g}$ is a Riemannian metric, and $\sqrt{g(\vec{\beta},\vec{\beta})} < 1$.

Note that: (for $\lambda > 0$)

$$R(\lambda v) = \sqrt{g(\lambda v,\lambda v)} + g(\lambda v,\vec{\beta}) = \lambda(\sqrt{g(v,v)}) + \lambda(g(v,\vec{\beta})) = \lambda R(v)$$

$$R(v) = \sqrt{g(v,v)} + g(v,\vec{\beta}) \neq R(-v) = \sqrt{g(v,v)} - g(v,\vec{\beta})$$

**Definition 4 (Randers Norm)**

$R(v) = \sqrt{g(v,v)} + g(v,\vec{\beta})$ such that $\mathrm{g}$ is a Riemannian metric, and $\sqrt{g(\vec{\beta},\vec{\beta})} < 1$.

Note that: (for $\lambda > 0$)

$$R(\lambda v) = \sqrt{g(\lambda v, \lambda v)} + g(\lambda v, \vec{\beta}) = \lambda(\sqrt{g(v,v)}) + \lambda(g(v,\vec{\beta})) = \lambda R(v)$$

$$R(v) = \sqrt{g(v,v)} + g(v,\vec{\beta}) \neq R(-v) = \sqrt{g(v,v)} - g(v,\vec{\beta})$$

**Definition 5 (Finsler Norm)**

A Manifold $M$ is called a **Finsler Manifold** if for each tangente space exists a *Minkowski norm*:

(1) $F : TM \to \mathbb{R}$ with $F(\lambda v) = \lambda F(v)$ for $\lambda > 0$;

(2) $g_v(u,w) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(v + su + tw)|_{s=t=0}$, is a metric.

# Distance and Geodesics

Let $\| \cdot \|$ be a Riemannian norm $\sqrt{\mathrm{g}(\cdot, \cdot)}$ or Finsleriaan norm $F(\cdot)$. **The distance between two points** is defined as:

$d(p, q) = \inf \sum_i \int_{t_i}^{t_i+1} \|\alpha_i'(t)\| dt$ such that $\alpha : [0, 1] \to M$ is a piece-wise smooth curve such that $\alpha(0) = p$ and $\alpha(1) = q$.

**Note that** in the Randers case $d(p, q) \neq d(q, p)$.

# Distance and Geodesics

Let $\|\cdot\|$ be a Riemannian norm $\sqrt{g(\cdot,\cdot)}$ or Finsleriaan norm $F(\cdot)$. **The distance between two points** is defined as:

$d(p,q) = \inf \sum_i \int_{t_i}^{t_i+1} \|\alpha_i'(t)\| dt$ such that $\alpha : [0,1] \to M$ is a piece-wise smooth curve such that $\alpha(0) = p$ and $\alpha(1) = q$.

**Note that** in the Randers case $d(p,q) \neq d(q,p)$.

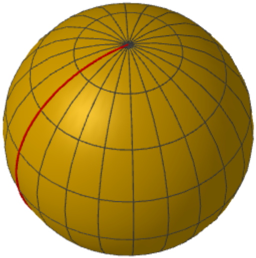
Definition 6 (Geodesic)

A smooth curve parameterized by arc-length $\gamma : I \to M$ is called a **geodesic** if it *minimizes locally the distance*, i.e, for each $s_0 \in I$ exists $\epsilon > 0$ such that $d(\gamma(s_0), \gamma(s)) = \int_{s_0}^{s} \|\gamma'(t)\| dt = s - s_0$ for $s \in [s_0, s_0 + \epsilon]$.

# Distance and Geodesics

**Definition 6 (Geodesic)**

A smooth curve parameterized by arc-length $\gamma : I \to M$ is called a **geodesic** if it *minimizes locally the distance*, i.e, for each $s_0 \in I$ exists $\epsilon > 0$ such that $d(\gamma(s_0), \gamma(s)) = \int_{s_0}^{s} \|\gamma'(t)\| dt = s - s_0$ for $s \in [s_0, s_0 + \epsilon]$.



# Newton Equation

**Proposition 7 (Jacobi Metric)**

*Let $\gamma$ be a solution to the Newton Equation $(\gamma''(t))^T = -(\nabla U)^T$ at a Riemannian Manifold with Riemannian induced metric $(M, g_0)$ such that the potential function $U$ is bounded above $(U < c)$. Then $\gamma$ is a geodesic of the Jacobi metric $(c - U)g_0$ up to reparametrization.*

# Randers Geodesics

Alternatively, the Randers norm can be defined as:

$$R(v) = h(v - R(v)W)$$

where $h$ is a **Riemannian norm** and $W$ is a vector field such that $h(W, W) < 1$(called **Wind**).

The pair $(h, W)$ is called **Zermelo data**, on $T_pM$:

$$B^R_\epsilon(0) = B^h_\epsilon(0) + \epsilon W_p$$

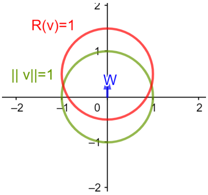
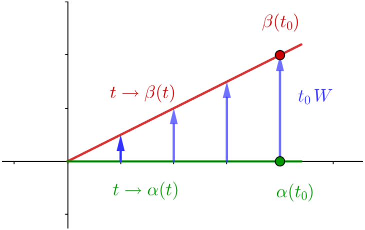



Figura: Vide S. Markvorsen, *A Finsler geodesic spray paradigm for wildfire spread modelling*, Nonlinear Anal., Real World Appl. 28 (2016) 208–228.

## Theorem 8 (1)

*Consider a Randers norm $R$ with Zermelo data $(h, W)$ in $M$, such that $W$ is a Killing Vector Field (i.e., a vector field whose flow is and isometry). Let $\alpha : I \subset \mathbb{R} \to M$ be a arc-length parametrized geodesic in the Riemannian manifold $(M, h)$.*
*Then $\beta(t) = \varphi_t(\alpha(t))$ is a Randers arch-length parametrized geodesic, such that $\varphi_t$ is the flow of $W$.*





[1]: P. Foulon and V. S. Matvee, Zermelo Deformation of Finsler metrics, Electronic Research Announcements, V. 25 (2018), 1–7

## Katok's Example on $S^2$

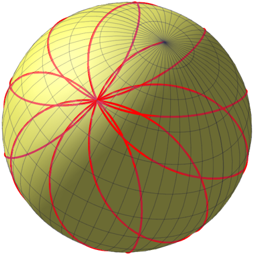


Figura: Randers Geodesic on the sphere, obtained from the Zermelo data $(h, W)$ where $h$ is the Euclidean metric on the sphere $S^2$ and $W$ is a rotation with irrational angular velocity of $\frac{1}{\sqrt{5}}$

# Distance Function and Forward Parallelism

Given $f : M \to \mathbb{R}$ a partition $\mathcal{F} = \{f^{-1}(c)\}$ is called **forward parallel** if for $c_0 < c_1$

$$x \in f^{-1}(c_1) \cap C_\epsilon^+(f^{-1}(c_0)) \Longrightarrow f^{-1}(c_1) \subset C_\epsilon^+(f^{-1}(c_0))$$

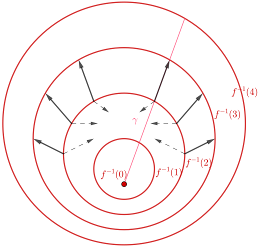



Figura: $f(x) = d(0, x)$

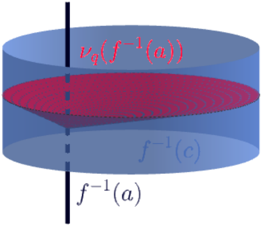



Figura: $f(x) = d(0, x)$

## Theorem 9 (3)

*Wavefronts are pre-images of distance functions relatively to an source A. In other words, for $f(x) = d(A, x)$ the wavefronts are $f^{-1}(c)$.*

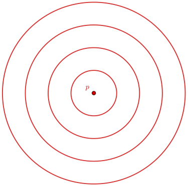


Figura: Huygens: *each point of a wavefront functions as a new punctual source.*

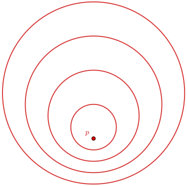


Figura: Markvorsen: Model for wildfire (with mild wind)

[3] H.R. Dehkordi and S. Alberto, *Huygens? envelope principle in Finsler spaces and analogue gravity.* Classical and Quantum Gravity 36.8 (2019): 085008.

# Transnormal Function: generalizing the distance function

**Definition 10**

Given a Finsler manifold $(M, F)$, a smooth function $f : M \to \mathbb{R}$ is called $F$-**transnormal function** if:

$$F(\nabla f)^2 = b(f)$$

where $b$ is continuous.

- $f(x, y) = \sqrt{x^2 + y^2} \to \|\nabla f(x, y)\|^2 = 1$ , then $b \equiv 1$.
- $f(x, y) = x^2 + y^2 \to \|\nabla f(x, y)\|^2 = 4(x^2 + y^2)$, then $b(z) = 4z$.

# Transnormal Function: generalizing the distance function

**Definition 10**

Given a Finsler manifold $(M, F)$, a smooth function $f : M \to \mathbb{R}$ is called $F$-**transnormal function** if:

$$F(\nabla f)^2 = b(f)$$

where $b$ is continuous.

- $f(x, y) = \sqrt{x^2 + y^2} \to \|\nabla f(x, y)\|^2 = 1$ , then $b \equiv 1$.
- $f(x, y) = x^2 + y^2 \to \|\nabla f(x, y)\|^2 = 4(x^2 + y^2)$, then $b(z) = 4z$.

**Remark 11**

- $df(\cdot) = g_{\nabla f}(\nabla f, \cdot)$

**Question:** *Under wich conditions the level sets are forward and backward parallel, in other words,* $\mathcal{F} = \{f^{-1}(c)\}$ *is a Finsler Partition?*

**Question:** *Under wich conditions the level sets are forward and backward parallel, in other words, $\mathcal{F} = \{f^{-1}(c)\}$ is a Finsler Partition?*

### Theorem 12

*Let $(M, F)$ be a analytic, compact, connected Finsler Manifold and $f : M \to \mathbb{R}$ an $F$-transnormal analytic function with $f(M) = [a, b]$. Suppose that the level sets are connected sets and $a, b$ are the only singular values on $[a, b]$. Then:*

(a) *The sets $f^{-1}(a)$ and $f^{-1}(b)$ are submanifolds.*

(b) *The level sets are equidistant, in other words, $\mathcal{F} = \{f^{-1}(c)\}$ is a Finsler Foliation.*

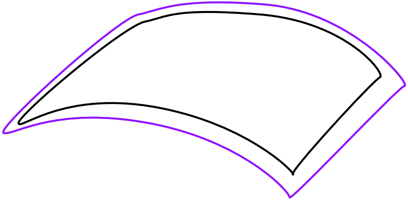
[2] M. M. Alexandrino, B. O. Alves, H. R. Dehkordi, *On Finsler transnormal functions*, Differential Geometry and its Applications Volume 65, 93-107 (2019)

# Newton Equation

**Proposition 13 (Jacobi Metric)**

*Let $\gamma$ be a solution to the Newton Equation $(\gamma''(t))^T = -(\nabla U)^T$ at a Riemannian Manifold with Riemannian induced metric $(M, g_0)$ such that the potential function $U$ is bounded above $(U < c)$. Then $\gamma$ is a geodesic of the Jacobi metric $(c - U)g_0$ up to reparametrization.*

# Sketch of the Hamiltonian proof of the Jacobi Metric Theorem



$$H_1^{-1}(c_1) = H_2^{-1}(c_2)$$

- Then, the symplectic gradients are multiples of one another, it means: $X_{H_2}(z) = \lambda(z) X_{H_1}(z)$

- There exists $\varphi$, such that if $\alpha_1, \alpha_2$ are solutions to the flows of $X_{H_1}, X_{H_2}$ respectively, $\alpha_2 = \alpha_1 \circ \varphi$.

- Apply the result to the following Hamiltonians:

$$H(v_p) = \frac{1}{2}\|v_p\|^2 + U(p)$$

$$H_J(v_p) = \frac{\|v_p\|^2}{2(c - U(p))}$$

# Sketch of the Lagrangian proof of the Jacobi Metric Theorem

- Note that solutions of the *Newton Equantion* have constant energy.

- Compare the *Levi-Civita Connections* of the metrics $g$ and $g_J$ using *Koszul's Formula*.

- Remember that by the chain rule, if $h$ is a reparametrization and $\beta = \alpha \circ h$, then $\frac{\nabla}{dt}\beta'(t) = h''(t)\alpha'(h(t)) + (h'(t))^2 \frac{\nabla}{dt}\alpha'(h(t))$.

- Conclude the existence of a reparametrization $h$ that turn solutions of Euler-Lagrange Equation of one Lagrangian to solutions of the other one.

# Sketch of the proof of the Theorem on Randers Geodesics

- From properties of the flow $\varphi$ and the chain rule we calculate that the derivative of $\beta$ is: $\beta'(t) = W(\varphi_t(\alpha(t))) + \varphi_{t*}\alpha'(t)$.

# Sketch of the proof of the Theorem on Randers Geodesics

- From properties of the flow $\varphi$ and the chain rule we calculate that the derivative of $\beta$ is: $\beta'(t) = W(\varphi_t(\alpha(t))) + \varphi_{t*}\alpha'(t)$.
- As $\varphi$ preserves $h$ and $W$, it can be proven that it preserves $R$. Then:

$$R(W(\varphi_t(\alpha(t))) + \varphi_{t*}\alpha'(t)) = R(W(\alpha(t)) + \alpha'(t)) = h(\alpha'(t))$$

# Sketch of the proof of the Theorem on Randers Geodesics

- From properties of the flow $\varphi$ and the chain rule we calculate that the derivative of $\beta$ is: $\beta'(t) = W(\varphi_t(\alpha(t))) + \varphi_{t*}\alpha'(t)$.
- As $\varphi$ preserves $h$ and $W$, it can be proven that it preserves $R$. Then:

$$R(W(\varphi_t(\alpha(t))) + \varphi_{t*}\alpha'(t)) = R(W(\alpha(t)) + \alpha'(t)) = h(\alpha'(t))$$

- Then, $\beta$ is arch-length parametrized relatively to $R$ and the following equality is true:

$$\int h(\alpha'(t))dt = \int R(\beta'(t))dt$$

# Sketch of the proof of the Theorem on Randers Geodesics

- From properties of the flow $\varphi$ and the chain rule we calculate that the derivative of $\beta$ is: $\beta'(t) = W(\varphi_t(\alpha(t))) + \varphi_{t*}\alpha'(t)$.
- As $\varphi$ preserves $h$ and $W$, it can be proven that it preserves $R$. Then:

$$R(W(\varphi_t(\alpha(t))) + \varphi_{t*}\alpha'(t)) = R(W(\alpha(t)) + \alpha'(t)) = h(\alpha'(t))$$

- Then, $\beta$ is arch-length parametrized relatively to $R$ and the following equality is true:

$$\int h(\alpha'(t))dt = \int R(\beta'(t))dt$$

- As $\alpha$ is a geodesic, it minimizes locally the distance, then:

$$d_R(0, \varphi_\epsilon(p)) \leqslant d_h(0, p)$$

# Sketch of the proof of the Theorem on Randers Geodesics

- From properties of the flow $\varphi$ and the chain rule we calculate that the derivative of $\beta$ is: $\beta'(t) = W(\varphi_t(\alpha(t))) + \varphi_{t*}\alpha'(t)$.
- As $\varphi$ preserves $h$ and $W$, it can be proven that it preserves $R$. Then:

$$R(W(\varphi_t(\alpha(t))) + \varphi_{t*}\alpha'(t)) = R(W(\alpha(t)) + \alpha'(t)) = h(\alpha'(t))$$

- Then, $\beta$ is arch-length parametrized relatively to $R$ and the following equality is true:

$$\int h(\alpha'(t))dt = \int R(\beta'(t))dt$$

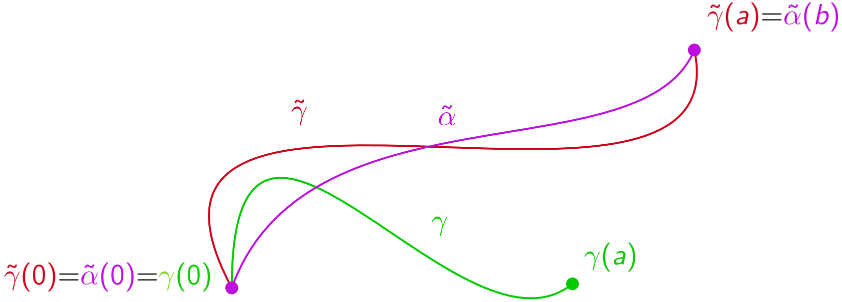
- As $\alpha$ is a geodesic, it minimizes locally the distance, then:

$$d_R(0, \varphi_\epsilon(p)) \leqslant d_h(0, p)$$

- A similar argument is constructed, consider $h$ a norm with data $(R, -W)$ and it is obtained that $\beta$ minimizes locally the Randers distance, hence it is a geodesic.

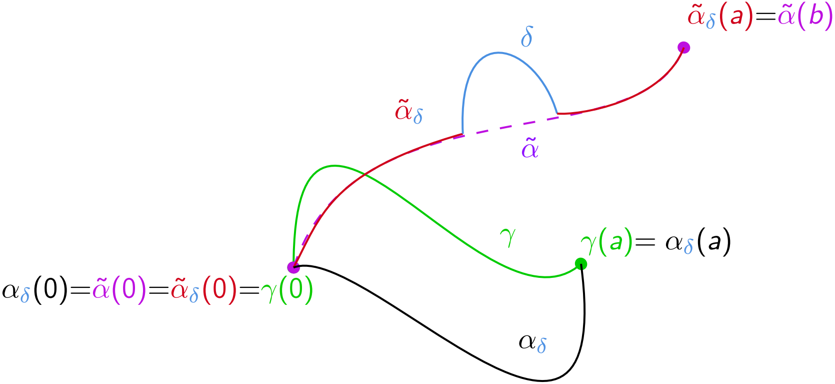
# Sketch of the proof of the Theorem on Randers Geodesics

- A similar argument is constructed, consider $h$ a norm with data $(R, -W)$ and it is obtained that $\beta$ minimizes locally the Randers distance, hence it is a geodesic.



# Sketch of the proof of the Theorem on Randers Geodesics

- A similar argument is constructed, consider $h$ a norm with data $(R, -W)$ and it is obtained that $\beta$ minimizes locally the Randers distance, hence it is a geodesic.



# Sketch of the proof of the Theorem on Foliations

- Use that pre-image of real analytic functions are a stratification.
- Using compactness, analyticity and the codimension 1 it can be proven that exists a neighborhood without the singular leaves that the partition is Finsler. To prove this one analyses the integral curves of $\nabla f$.
- Using the compactness and analyticity again it can be proven that the derivative of the endpoint map $\eta_{t\xi}$ has constant rank in each leaf. Using the Rank Theorem and tubular neighborhood arguments one proves that the singular leafs are submanifolds.
- Using analyticity and compactness to analyze the integral curves of $\nabla f$ and the tubular neighborhood argument one extends the properties to the whole manifold, finishing the proof.

# Main Bibliography


- Foulon, Patrick; Matveev, Vladimir. S.; *Zermelo Deformation of Finsler Metrics by Killing Vector Fields*, Electronic Research Announcements, **25** (2018), 1-7.
- Alexandrino, M. M.; Alves, B. O.; Dehkordi, H. R. *On Finsler transnormal functions* Differential Geometry and its Applications v 65 (2019) 93–107.
- Alves, Benigno O. *Sobre Folheações Finslerianas Singulares* Tese de Doutorado IME-USP 2017 (orientador Prof. M. Alexandrino) https://teses.usp.br/teses/disponiveis/45/45131/tde-14122017-130959/pt-br.php
- Gonçalves, Guilherme Casas; *Rudimento de Mecânica, Ações Hamiltoneanas e Aplicação Momento* Dissertação de Mestrado IME-USP 2015