MODULUS OF CONTINUITY OF SOLUTIONS TO COMPLEX MONGE-AMPÈRE EQUATIONS ON STEIN SPACES

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Abstract. In this paper, we study the modulus of continuity of solutions to Dirichlet problems for complex Monge-Ampère equations with L^p densities on Stein spaces with isolated singularities. In particular, we prove such solutions are Hölder continuous outside singular points if the boundary data is Hölder continuous.

Contents

Introduction

Let X be a Stein complex space that is reduced, locally irreducible, and of complex dimension $n \geqslant 1$ with an isolated singularity, where $X_{\text{sing}} = \{0\} \subset \mathbb{C}^N$. Equip X with a Hermitian metric whose fundamental form is β, a positive $(1,1)$-form, and let $d_\beta(\cdot,\cdot)$ be the distance induced by β. Let Ω be a bounded, strongly pseudoconvex open subset of X, and fix ρ as a smooth, strictly plurisubharmonic defining function, i.e., $\Omega = \{\rho < 0\}$. Given $\phi \in C^0(\partial\Omega)$ and $f \in L^p(\Omega, \beta^n)$ with $p > 1$, we consider the Dirichlet problem:

$$MA(\Omega, \phi, f) : \begin{cases} (dd^c u)^n = f\beta^n & \text{in} \quad \Omega, \\ u = \phi & \text{on} \quad \partial\Omega, \end{cases}$$

where $d = \partial + \overline{\partial}$ and $d^c = i(\overline{\partial} - \partial)$. This Dirichlet problem for the complex Monge-Ampère operator has been intensively studied in domains of $\mathbb{C}^n$ (see [GZ17] for a historical account).

The study of complex Monge-Ampère equations is key to understanding canonical metrics in Kähler manifolds. This significance extends to mildly singular varieties as well. Interactions with birational geometry, such as the Minimal Model Program, make it enticing across fields to comprehend these equations.

In this paper, our objective is to explore the impact of boundary data and singularity types on the modulus of continuity of solutions. In particular, we prove that if ϕ is *Hölder continuous*, then so is the solution of $MA(\Omega, \phi, f)$ outside the singular point.

Regarding applications of the Hölder regularity of solutions to complex Monge-Ampère equations, we refer to [GGZ23c] for some geometric consequences of the Hölder continuity

Date: May 28, 2024.

of Kähler-Einstein potentials and to [DNS10] for more details about applications in complex dynamics.

Several works have focused on the Hölder continuity of solutions, both on domains of $\mathbb{C}^n$ (see [GKZ08, Cha15, Cha17, BKPZ16]) and on compact complex manifolds (see [DDGHKZ14, KN18, LPT21, DKN22]).

However, there are only a few literature dealing with this problem on domains with singularities. Recently, it was proved in [GGZ23a] that the solution $u = u(\Omega, \phi, f)$ for $MA(\Omega, \phi, f)$ is continuous and unique in $\overline{\Omega}$. Further smoothness properties have been provided in [DFS23, F23, GT23] on the regular part of domains that contain an isolated singularity in the case of smooth boundary data.

Our main result is the following:

Theorem A. *Assume that $\phi \in C^0(\partial\Omega)$ and $f \in L^p(\Omega, \beta^n)$ with $p > 1$. Then the unique solution $u \in PSH(\Omega) \cap C^0(\overline{\Omega})$ to $MA(\Omega, \phi, f)$ has the following modulus of continuity at $x \in \Omega$:*

$$\omega_{u,x}(t) \leqslant C_x \max\{\omega_\phi(t^{1/2}), t^{\frac{1}{nq+1}}\}$$

for some constant $C_x > 0$ such that $C_x \to +\infty$ as $x \to 0$, where $\omega_{u,x}$ is the modulus of continuity of u at the point x and $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, Theorem A implies:

Corollary B. *When $\phi \in C^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leqslant 1$, and $f \in L^p(\Omega, \beta^n)$ with $p > 1$, the unique solution $u \in PSH(\Omega) \cap C^0(\overline{\Omega})$ to $MA(\Omega, \phi, f)$ is α^*-Hölder continuous outside the singular point, for $\alpha^* < \min\{\frac{\alpha}{2}, \frac{1}{nq+1}\}$.*

Remark. *This Hölder exponent*[1] *is the same for smooth domains* [Cha17, Theorem 0.2].

Remark. *One can similarly treat the case when X has finitely many isolated singularities.*

Acknowledgements. The author would like to first thank his PhD advisor, Vincent Guedj, for his support, guidance and comments. The author also thanks Ahmed Zeriahi and Chung-Ming Pan for useful discussions and references. This work received support from the University Research School EUR-MINT (State support managed by the National Research Agency for Future Investments program bearing the reference ANR-18-EURE-0023).

1. Preliminaries

1.1. **Complex analysis on Stein spaces.** Throughout this paper, we let X be a reduced, locally irreducible complex analytic space of pure dimension $n \geqslant 1$. We denote by X_{reg} the complex manifold of regular points of X and $X_{\mathrm{sing}} := X \setminus X_{\mathrm{reg}}$ the set of singular points, which is an analytic subset of X with complex codimension $\geqslant 1$.

By definition, for each point $x_0 \in X$, there exists a neighborhood U of x_0 and a local embedding $j : U \hookrightarrow \mathbb{C}^N$ onto an analytic subset of $\mathbb{C}^N$ for some $N \geqslant 1$. These local embeddings allow us to define the spaces of smooth forms of given degree on X as smooth forms on X_{reg} that are locally on X restrictions of an ambient form on $\mathbb{C}^N$. Other differential notions and operators, such as holomorphic and plurisubharmonic functions, can also be defined in this way. (See [Dem85] for more details). Two different notions can be defined for plurisubharmonicity:

[1]Relative to the induced distance of the chosen β.

Definition 1.1. *Let $u : X \to \mathbb{R} \cup \{-\infty\}$ be a given function.*

(1) *We say that u is plurisubharmonic (psh for short) on X if it is locally the restriction of a plurisubharmonic function in a local embedding of X.*
(2) *We say that u is weakly plurisubharmonic on X if u is locally bounded from above on X, and its restriction to the complex manifold X_{reg} is plurisubharmonic.*

Fornæss and Narasimhan [FN80, Theorem 5.3.1] proved that u is plurisubarmonic on X if and only if, for any analytic disc $h : \mathbb{D} \to X$, the function $u \circ h$ is subharmonic or identically $-\infty$. If u is weakly plurisubharmonic on X, then u is plurisubharmonic on X_{reg}, and thus upper semi-continuous on X_{reg}. It is natural to extend u to X using the following formula:

$$u^*(x) := \limsup_{X_{\mathrm{reg}} \ni y \to x} u(y), \quad x \in X. \tag{1.1}$$

The function u^* is upper semi-continuous, locally integrable on X, and satisfies $dd^c u^* \geqslant 0$ in the sense of currents on X [Dem85, Théorème 1.7]. The two notions are equivalent when X is locally irreducible, as shown in [Dem85, Théorèm 1.10]:

Theorem 1.2. *Let X be a locally irreducible analytic space and $u : X \to \mathbb{R} \cup \{-\infty\}$ be a weakly plurisubharmonic function on X. Then the function u^* defined by* (1.1) *is psh on X.*

Note that since u is plurisubharmonic on X_{reg}, we have $u^* = u$ on X_{reg}. Then u^* is the upper semi-continuous extension of $u|_{X_{\mathrm{reg}}}$ to X. With this, we have the following:

Corollary 1.3. *Let $\mathcal{U} \subset \mathrm{PSH}(X)$ be a non-empty family of plurisubharmonic functions which is locally bounded from above on X. Then its upper envelope*

$$U := \sup\{u; u \in \mathcal{U}\}$$

is a well-defined Borel function whose upper semi-continuous regularization U^* is psh on X.

Recall that from [FN80, Theorem 6.1] X is Stein if it admits a C^2 strongly plurisubharmonic exhaustion. We will use the following definition:

Definition 1.4. *A domain $\Omega \Subset X$ is strongly pseudoconvex if it admits a negative smooth, strongly plurisubharmonic defining function, i.e., a strongly plurisubharmonic function ρ in a neighborhood Ω' of $\overline{\Omega}$ such that $\Omega := \{x \in \Omega'; \rho(x) < 0\}$ and for any $c < 0$,*

$$\Omega_c := \{x \in \Omega'; \rho(x) < c\} \Subset \Omega$$

is relatively compact.

We denote by $\mathrm{PSH}(X)$ the set of plurisubharmonic functions on X.

On complex spaces, the complex Monge-Ampère operator has been defined and studied in [Bed82] and [Dem85]. In this setting, if $u \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$, the Monge-Ampère measure $(dd^c u)^n$ is well defined on the regular part X_{reg} and can be extended to X as a Borel measure with zero mass on X_{sing}. This notion extends the foundational work of pluripotential theory by Bedford-Taylor [BT76, BT82].

Consequently, several standard properties of the complex Monge-Ampère operator acting on $\mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$ extend to this setting (see [Bed82, Dem85]). In particular, we have the following comparison principle [Bed82, Theorem 4.3]:

Proposition 1.5 (Comparison principle)**.** *Let $\Omega \Subset X$ be a relatively compact open set and $u, v \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$. Assume that $\liminf_{x \to \zeta}(u(x) - v(x)) \geqslant 0$ for any $\zeta \in \partial\Omega$. Then*

$$\int_{\{u<v\}} (dd^c v)^n \leqslant \int_{\{u<v\}} (dd^c u)^n .$$

In particular, if $(dd^c u)^n \leqslant (dd^c v)^n$ weakly on Ω, then $v \leqslant u$ on Ω.

1.2. **Notations.** We fix the following notations throughout this paper:

- X is a Stein space that is reduced and locally irreducible of complex dimension $n \geqslant 1$ with an isolated singularity identified as $X_{\text{sing}} = \{0\} \subset \mathbb{C}^N$.
- β is a fixed smooth positive (1,1)-form on X, the fundamental form of a Hermitian metric on X.
- d_τ is the distance induced by a fundamental form τ on some manifold (Y, τ). When no confusion can occur d_β will be written as $d(\cdot,\cdot)$.
- $\Omega \Subset X$ is a bounded, strongly pseudoconvex open subset of X and fix ρ smooth strictly psh defining function, i.e., $\Omega = \{\rho < 0\}$.
- $0 \leqslant f \in L^p(\Omega, \beta^n)$, $p > 1$ and $1/p + 1/q = 1$ and $\phi \in C^0(\partial\Omega)$.
- $u(\Omega, \phi, f)$ is the solution to the Dirichlet problem:
$$MA(\Omega, \phi, f) : \begin{cases} (dd^c u)^n = f\beta^n & \text{in} \quad \Omega, \\ u = \phi & \text{on} \quad \partial\Omega. \end{cases}$$
- $\omega_{g,x}$ is the modulus of continuity of some function g at the point x.
- $\lambda(x) = d(x, 0)$ the distance of a point x to the singular point. The point x will be omitted when there is no chance of confusion.
- $\mathbb{B}_{2n}$ is the volume of unit ball in $\mathbb{C}^n$. dV is the standard euclidian volume form in $\mathbb{C}^n$. $d_{\mathbb{C}^n}$ is the standard euclidian distance in $\mathbb{C}^n$.

1.3. **Useful Results.** We need the stability estimate from [GGZ23a, Proposition 1.8]:

Theorem 1.6. *Let φ, ψ be two bounded plurisubharmonic functions in Ω and $(dd^c\varphi)^n = f\beta^n$ in Ω. Fix $0 \leqslant \gamma < \frac{1}{nq+1}$. Then there exists a uniform constant $C = C(\gamma, ||f||_{L^p(\Omega)}) > 0$ such that:*
$$\sup_{\Omega}(\psi - \varphi) \leqslant \sup_{\partial\Omega} \, (\psi - \varphi)^*_+ + C(||(\psi - \varphi)_+||_{L^1(\Omega,\beta^n)})^\gamma$$
where $(\psi - \varphi)_+ := \max(\psi - \varphi, 0)$ and w^* is the upper semi-continuous extension of a bounded function w on Ω to the boundary, i.e., $w^*(\xi) := \limsup_{z\to\xi} w(z)$.

The local theory [Cha17, Lemmas 3.2 and 3.3] will be used to prove Lemma 4.1. Here reformulated to:

Theorem 1.7. *Let $\widehat{\Omega} \Subset \mathbb{C}^n$ strongly pseudoconvex domain, $0 \leqslant \widehat{f} \in L^p(\widehat{\Omega}); p > 1$ and $\varepsilon > 0$ small enough. For $\widehat{u} \in PSH(\widehat{\Omega}) \cap C^0(\overline{\widehat{\Omega}})$ such that $(dd^c\widehat{u})^n = \widehat{f}dV$ in $\widehat{\Omega}$. We have*
$$||\Lambda_{\delta/2}\widehat{u} - \widehat{u}||_{L^1(\widehat{\Omega}_\delta)} \leqslant C\delta^{1-\varepsilon}$$
for some constant $C = C(n, \varepsilon, \widehat{\Omega}, ||\widehat{u}||_{L^\infty(\Omega)}) > 0$, where $\widehat{\Omega}_\delta := \{z \in \widehat{\Omega} | d_{\mathbb{C}^n}(z, \partial\widehat{\Omega}) > \delta\}$ and for $z \in \widehat{\Omega}_\delta$, $\Lambda_\delta\widehat{u}(z) = \frac{1}{\mathbb{B}_{2n}\delta^{2n}} \int_{|z-\zeta|\leqslant\delta} \widehat{u}(\zeta)dV(\zeta)$ is the mean volume regularizing function in $\mathbb{C}^n$.

Moreover, if $\Delta\widehat{u}$ has finite mass in $\widehat{\Omega}$ then one can get δ^2 instead of $\delta^{1-\varepsilon}$.

We will also use, in the proof of Lemma 2.3, the following classical result from singular Riemannian foliations in [W87, Lemma 1], here reformulated to:

Lemma 1.8. *Let (M, g) be a smooth, connected, complete manifold with Riemannian metric g and induced distance function d.Fix $f \in C^2(M)$ such that $g(\nabla f, \nabla f) = 1$. Suppose $[a, b] \subset f(M) \subset \mathbb{R}$ contains no critical points of f. Then for any $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$ we have:*
$$d(x, f^{-1}(b)) = d(f^{-1}(a), y) = \int_a^b \mathrm{d}f = b - a.$$

2. Construction of barriers

In this section, we develop barrier functions, which allow us to control the modulus of continuity of solutions to Dirichlet problems close to the boundary. The barriers we will construct are described in the following proposition:

Proposition 2.1. *Let $u = u(\Omega, \phi, f)$ and fix $0 \leqslant \gamma < 1/(nq+1)$. Then there exist two barrier functions $v, w \in PSH(\Omega) \cap C^0(\overline{\Omega})$, such that:*

(1) $v(\xi) = \phi(\xi) = -w(\xi)$, $\forall \xi \in \partial\Omega$,
(2) $v(z) \leqslant u(z) \leqslant -w(z), \forall z \in \Omega$,
(3) $\omega_{v,x}(t), \omega_{w,x}(t) \leqslant C_x \max\{\omega_\phi(t^{1/2}), t^\gamma\}$,

for some constant $C_x > 0$ that goes to $+\infty$ as x approaches the singular point.

The proof is divided into the following three steps:

- **Step 1:** We treat the case where $\phi \equiv 0$ and f is bounded near $\partial\Omega$.
- **Step 2:** Construct barriers for the case where $f \equiv 0$.
- **Step 3:** Combine Steps 1 and 2 to solve the general case.

The construction of barriers depends on the behavior around the boundary. The idea of Step 1 comes from [GKZ08, Lemma 2.2] and Step 2 from [Cha15, Proposition 4.4].

2.1. Densities bounded near the boundary.

Lemma 2.2. *Assume that $\hat{f}$ is bounded near $\partial\Omega$ and set $u_0 := u(\Omega, 0, \hat{f})$. There exist $b_{\hat{f}}, w \in PSH(\Omega) \cap C^{0,1}(\Omega)$ such that:*

(1) $b_{\hat{f}}(\xi) = 0 = -w(\xi), \forall \xi \in \partial\Omega$,
(2) $b_{\hat{f}} \leqslant u_0 \leqslant -w$ *in* Ω.

Proof. Since $\hat{f}$ is bounded near $\partial\Omega$, there exists a compact $K \subset \Omega$ such that $0 \leqslant \hat{f} \leqslant M$ on $\Omega \setminus K$. Also, there exist $A_1, A_2 > 0$ large enough such that $(dd^c(A_1\rho))^n \geqslant M\beta^n \geqslant \hat{f}\beta^n$, on $\Omega \setminus K$ and $A_2\rho \leqslant m \leqslant u_0$ on a neighborhood of K, for $m := \min_{\overline{\Omega}} u_0$. Then, by taking the maximum of both constants, we have $A > 0$ large enough such that both conditions are satisfied for $b_{\hat{f}} := A\rho$, which is smooth plurisubharmonic on Ω. As we also have $b_{\hat{f}} \leqslant u_0$ on $\partial(\Omega \setminus K)$, by the comparison principle we get $b_{\hat{f}} \leqslant u_0$ in $\Omega \setminus K$. By construction, we have $b_{\hat{f}} \leqslant u_0$ on a neighborhood of K; hence, with the above argument we get $b_{\hat{f}} \leqslant u_0$ in Ω and $b_{\hat{f}}(\partial\Omega) = \{0\}$, thus is a lower-barrier. For an upper barrier, take $w \equiv 0$ in Ω as by the maximum principle $u_0 \leqslant 0$ and it is zero on the boundary. □

2.2. Zero density.

Lemma 2.3. *Let $u_\phi := u(\Omega, \phi, 0)$. There exists a barrier $h_\phi \in PSH(\Omega) \cap C^0(\overline{\Omega})$ such that:*

(1) $h_\phi(\xi) = \phi(\xi) = -h_{-\phi}(\xi), \forall \xi \in \partial\Omega$,
(2) $h_\phi \leqslant u_\phi, \leqslant -h_{-\phi}$ *in* Ω,
(3) $\omega_{h_\phi}(t) \leqslant C\omega_\phi(t^{1/2})$.

Proof. We set $h_\phi \in PSH(\Omega) \cap C^0(\overline{\Omega})$ such that:

$$\begin{cases} \omega_{h_\phi}(t) \leqslant C\omega_\phi(t^{1/2}) & \text{in} \quad \Omega, \\ h_\phi = \phi & \text{on} \quad \partial\Omega. \end{cases}$$

With such $h_\phi \in PSH(\Omega)$ we have $(dd^c h_\phi)^n \geqslant 0 = (dd^c u_\phi)^n$. Hence, by the comparison principle, we get $h_\phi \leqslant u_\phi$. Similarly, we get $-h_{-\phi} \leqslant -u_\phi$; hence $h_{-\phi} \geqslant u_\phi$. Now we start the construction of such h_ϕ:

Step 1: *For fixed $\xi \in \partial\Omega$, construct a function h_ξ close to ξ.*

Fix $\xi \in \partial\Omega$. We want to find a function $h_\xi \in PSH(\Omega) \cap C^0(\overline{\Omega})$ such that:

$$\begin{cases} \omega_{h_\xi}(t) \leqslant C\omega_\phi(t^{1/2}) & , \\ h_\xi(x) \leqslant \phi(x) & \text{for any } x \in \partial\Omega, \\ h_\xi(\xi) = \phi(\xi) & . \end{cases}$$

In this first step, we will construct h_ξ above just around ξ. Take $B > 0$ large enough such that: $g(x) := B\rho(x) - (d_\beta(x,\xi))^2$ is in $PSH(\Omega)$.

Consider $\overline{\omega}_\phi$, the minimal concave majorant of ω_ϕ, and define $\chi(t) := -\overline{\omega}_\phi((-t)^{1/2})$, which is a convex non-decreasing function on $[-d^2, 0]$, where $d = diam(\Omega)$, the diameter of Ω.

Fix $r > 0$ sufficiently small such that $|g(x)| \leqslant d^2$ in $B_r(\xi) \cap \Omega$ and define for $x \in B_r(\xi) \cap \overline{\Omega}$: $\widetilde{g}(x) = \chi(g(x)) + \phi(\xi) = -\overline{\omega}_\phi[((d_\beta(x,\xi))^2 - B\rho(x))^{1/2}] + \phi(\xi)$. Note that $\widetilde{g}$ is a continuous psh function on $B_r(\xi) \cap \Omega$.

We have $\forall x \in \partial\Omega \cap B_r(\xi)$, $\phi(\xi) - \overline{\omega}_\phi(d_\beta(x,\xi)) \leqslant \phi(x)$. However, by our construction, $\widetilde{g}(x) = \phi(\xi) - \overline{\omega}_\phi(d_\beta(x,\xi))$ for any $x \in \partial\Omega \cap B_r(\xi)$. Hence, $\widetilde{g}(x) \leqslant \phi(x), \forall x \in \partial\Omega \cap B_r(\xi)$ and $\widetilde{g}(\xi) = \phi(\xi)$. Using the subadditivity of $\overline{\omega}_\phi$ and also the fact that $\forall t, \lambda > 0; \omega_\phi(\lambda t) \leqslant \overline{\omega}_\phi(\lambda t) \leqslant (1+\lambda)\omega_\phi(t)$ (see [Cha15, Lemma 4.1]) we get:

$$\begin{aligned} \omega_{\widetilde{g}}(t) &= \sup_{d_\beta(x,y)\leqslant t} |\widetilde{g}(x) - \widetilde{g}(y)| \\ &\leqslant \sup_{d_\beta(x,y)\leqslant t} \overline{\omega}_\phi[((d_\beta(x,\xi))^2 - (d_\beta(y,\xi))^2 - B(\rho(x) - \rho(y)))^{1/2}] \\ &\leqslant \sup_{d_\beta(x,y)\leqslant t} \overline{\omega}_\phi[(2d + B_1)^{1/2}(d_\beta(x,y))^{1/2}] \\ &\leqslant (1 + (2d + B_1)^{1/2}) \sup_{d_\beta(x,y)\leqslant t} \omega_\phi[(d_\beta(x,y))^{1/2}] \\ &\leqslant (1 + (2d + B_1)^{1/2}) \sup_{d_\beta(x,y)\leqslant t} \omega_\phi[(t)^{1/2}]. \end{aligned}$$

From the second line to the third, there are the following arguments: the first term goes as $d_1^2 - d_2^2 = (d_1 - d_2)(d_1 + d_2)$. Then, the first difference of distances can be bounded by $d_\beta(x,y)$ by triangular inequality. Also, the second term can be bounded by $2d$. Finally, the last term comes from the fact that, close to the boundary, we have that $|\rho(x) - \rho(y)|$ behaves like $d_\beta(x,y)$, by Lemma 1.8 up to renormalization of the gradient close to the boundary.

Step 2: *Extending our local $\widetilde{g}$ to all $\overline{\Omega}$ as h_ξ.*

Recall $\xi \in \partial\Omega$, fix $0 < r_1 < r$ and $\gamma_1 > \frac{d}{r_1}$ such that $\forall x \in \overline{\Omega} \cap \partial B_{r_1}(\xi)$:

$$-\gamma_1 \overline{\omega}_\phi[((d_\beta(x,\xi))^2 - B\rho(x))^{1/2}] = \gamma_1(\widetilde{g}(x) - \phi(\xi)) \leqslant \inf_{\partial\Omega} \phi - \sup_{\partial\Omega} \phi.$$

Set $\gamma_2 = \inf_{\partial\Omega} \phi$. Then:

$$\gamma_1(\widetilde{g}(x) - \phi(\xi)) + \phi(\xi) \leqslant \gamma_2 \tag{2.1}$$

for $x \in \partial B_{r_1}(\xi) \cap \overline{\Omega}$. Consider:

$$h_\xi(x) := \begin{cases} \max\{\gamma_1(\widetilde{g}(x) - \phi(\xi)) + \phi(\xi), \gamma_2\} & \textit{for } x \in B_{r_1}(\xi) \cap \overline{\Omega}, \\ \gamma_2 & \textit{for } x \in \overline{\Omega} \setminus B_{r_1}(\xi). \end{cases}$$

It follows from inequality (2.1) that h_ξ is a psh function on Ω, continuous on $\overline{\Omega}$ and such that $h_\xi(x) \leqslant \phi(x), \forall x \in \partial\Omega$ because on $\partial\Omega \cap B_{r_1}(\xi)$:

$$\gamma_1(\widetilde{g}(x) - \phi(\xi)) + \phi(\xi) = -\gamma_1\overline{\omega}_\phi[d_\beta(x,\xi)] + \phi(\xi) \leqslant -\overline{\omega}_\phi[d_\beta(x,\xi)] + \phi(\xi) \leqslant \phi(x).$$

Then finally, we have that h_ξ is a barrier relative to the point ξ, chosen generically, that is: for each $\xi \in \partial\Omega, \exists h_\xi \in S(\Omega,\phi,0); h_\xi(\xi) = \phi(\xi)$ and $\omega_{h_\xi}(t) \leqslant C\omega_\phi(t^{1/2})$; for $C = (1 + (2d + B_1)^{1/2})$ and $S(\Omega,\phi,0)$ is the set of subsolutions for the Dirichlet problem $MA(\Omega,\phi,0)$.

Step 3: *Take envelope in ξ to obtain the desired barrier h_ϕ.*

Now set $h_\phi(x) = \sup\{h_\xi(x) | \xi \in \partial\Omega\}$, which is a sup over a compact family; hence h_ϕ^* is a psh function. Note that $0 \leqslant \omega_{h_\phi}(t) \leqslant C\omega_\phi(t^{1/2})$; then $\omega_{h_\phi}(t) \to 0$ as $t \to 0$. This implies $h_\phi \in C^0(\overline{\Omega})$ and $h_\phi = h_\phi^* \in PSH(\Omega)$. (Hence $h_\phi \in S(\Omega,\phi,0)$.) By construction, $h_\phi = \phi$ on $\partial\Omega$. Thus, we have the desired lower-barrier for $MA(\Omega,\phi,0)$ with modulus of continuity $\omega_\phi(t^{1/2})$. □

2.3. The general case.

Lemma 2.4. *Let $\hat{f} \in L^p(\Omega,\beta^n), p > 1$ with $\hat{f}$ is bounded near $\partial\Omega$ and $0 \leqslant \gamma < \frac{1}{(nq+1)}$. Then $u(\Omega,\phi,0), u(\Omega,0,\hat{f})$ have modulus of continuity $C_x \max\{\omega_\phi(t^{1/2}), t^\gamma\}$ and $C_x t^{2\gamma}$ respectively, for $x \in \overline{\Omega}$ and some constant $C_x > 0$ that goes to $+\infty$ as $x \to 0$.*

Proof. Running the construction of this paper to the particular Dirichlet problems $MA(\Omega,\phi,0)$ and $MA(\Omega,0,\hat{f})$, since we already have their barriers, we get from Theorem 4.3 the solution $u(\Omega,\phi,0)$ will have modulus of continuity $\max\{\omega_\phi(t^{1/2}), t^\gamma\}$. Similarly, we get that $u(\Omega,0,\hat{f})$ will have modulus of continuity $t^{2\gamma}$, because the boundary data is zero. One gets this regularity by applying the proof of [Cha17, Proposition 2.1] to get that $\Delta u(\Omega,0,\hat{f})$ is bounded in Ω. Hence, we get the regularity $t^{2\gamma}$ out of Lemma 4.1. □

Now we prove our Proposition 2.1 for the solution of the general problem $u := u(\Omega,\phi,f)$:

Proof of Proposition 2.1: The upper-barrier will be the same as in Lemma 2.3 $w = u(\Omega,-\phi,0)$. Since $-w = u(\Omega,\phi,0)$, $-w = \phi$ in $\partial\Omega$ and $0 = (dd^c(-w))^n \leqslant (dd^c u)^n$, by the comparison principle we get $-w \geqslant u$ in Ω and by the Lemma 2.4 we know the modulus of continuity of w.

For the lower-barrier, we take a bigger pseudoconvex domain $\Omega \Subset \widehat{\Omega} \Subset X$ and extend trivially by zero the density f to $\widehat{\Omega}$, name it $\widehat{f}$. Note that $\widehat{f}$ is bounded near $\partial\widehat{\Omega}$. The lower-barrier will be $v := u(\widehat{\Omega},0,\widehat{f})|_\Omega + u(\Omega,\phi - u(\widehat{\Omega},0,\widehat{f})|_{\partial\Omega},0)$. By construction $v|_{\partial\Omega} = \phi$ and $(dd^c v)^n \geqslant f\beta^n + 0$ in Ω, by the comparison principle $v \leqslant u$ in Ω and by Lemma 2.4 we know the modulus of continuity of v. □

3. Regularization of the solution

This section provides an appropriate regularization scheme of the solution $u = u(\Omega,\phi,f)$ to $MA(\Omega,\phi,f)$, following ideas from [GKZ08, DDGHKZ14].

3.1. Defining regularization.

First, we define the set $\Omega_\delta := \{x \in \Omega \mid d_\beta(x,\partial\Omega) > \delta\}$ for $0 < \delta \leqslant \delta_0$ where δ_0 is fixed such that $\Omega_{\delta_0} \neq \emptyset$.

We consider $\pi : \widetilde{X} \to X$ a resolution of singularities of X. We will denote objects on $\widetilde{X}$ with a tilde. For example, $\widetilde{\Omega} = \{\pi^*\rho := \widetilde{\rho} < 0\}$. It is known that $\widetilde{X}$ is a Kähler manifold. Fix a Hermitian form τ on $\widetilde{X}$, assuming for simplicity[2] that τ and β closed.

Consider for each $\widetilde{x} \in \widetilde{X}$ the exponential map $\exp_{\widetilde{x}} : T_{\widetilde{x}}\widetilde{X} \ni \xi \mapsto \exp_{\widetilde{x}}(\xi) \in \widetilde{X}$ defined by $\exp_{\widetilde{x}}(\xi) = \sigma(1)$ with σ being the geodesic such that $\sigma(0) = \widetilde{x}$ and initial velocity $\sigma'(0) = \xi$. Let $\widetilde{u} := \pi^*u$ be the pull-back of the solution to $MA(\Omega, \phi, f)$. Define its δ-regularization $\eta_\delta\widetilde{u}$ as in [Dem82] by:

$$\eta_\delta\widetilde{u}(\widetilde{x}) = \frac{1}{\delta^{2n}} \int_{\xi \in T_{\widetilde{x}}\widetilde{X}} \widetilde{u}\left(\exp_{\widetilde{x}}(\xi)\right) \eta\left(\frac{|\xi|^2_\tau}{\delta^2}\right) dV_\tau(\xi), \quad \delta > 0 \text{ and } \widetilde{x} \in \widetilde{\Omega}_\delta.$$

Here η is a smoothing kernel, $|\xi|^2_\tau$ stands for $\sum_{i,j=1}^n g_{i\bar{j}}(\widetilde{x})\xi_i\bar{\xi}_j$, and $dV_\tau(\xi)$ is the induced measure $\frac{1}{2^n n!}\left(dd^c|\xi|^2_\tau\right)^n$.

It is known that:

$$\exp : T\widetilde{X} \to \widetilde{X}, \quad T\widetilde{X} \ni (\widetilde{x}, \xi) \mapsto \exp_{\widetilde{x}}(\xi) \in \widetilde{X}, \xi \in T_{\widetilde{x}}\widetilde{X}$$

has the following properties:

(1) exp is a C^∞ mapping;
(2) $\forall \widetilde{x} \in \widetilde{X}, \exp_{\widetilde{x}}(0) = \widetilde{x}$ and $D_\xi \exp(0) = \mathrm{Id}_{T_{\widetilde{x}}\widetilde{X}}$.

One can formally extend $\eta_\delta\widetilde{u}$ as a function on $\widetilde{\Omega}_\delta \times \mathbb{C}$ by putting $U(\widetilde{x}, w) := \eta_\delta\widetilde{u}(\widetilde{x})$ for $w \in \mathbb{C}$ with $|w| = \delta$. Coupling the estimate of the hessian of $U(\widetilde{x}, w)$ with Kiselman's minimum principle, one gets [DDGHKZ14, Lemma 2.1]:

Lemma 3.1. *For a bounded psh function $\widetilde{u}$ on the Kähler manifold $(\widetilde{\Omega} \Subset \widetilde{X}, \tau)$. Let $U(\widetilde{x}, w)$ be its regularization as defined above. Define the Kiselman-Legendre transform at level c by*

$$\widetilde{u}_{c,\delta}(\widetilde{x}) := \inf_{0 \leqslant t \leqslant \delta} \left[U(\widetilde{x}, t) + Kt^2 - K\delta^2 - c\log(t/\delta)\right]$$

for $\widetilde{x} \in \widetilde{\Omega}_\delta$ and some positive constant K depending on the curvature of $(\overline{\widetilde{\Omega}}, \tau)$ such that the function $U(\widetilde{x}, t) + Kt^2$ is increasing for $t \in (0, \delta_1)$ for some $0 < \delta_1 \leqslant \delta_0$ small enough. Also one has the following estimate for the complex hessian:

$$dd^c\widetilde{u}_{c,\delta} \geqslant -\left(Ac + K\delta\right)\tau$$

where A is a lower bound of the negative part of the bisectional curvature of $(\overline{\widetilde{\Omega}}, \tau)$.

3.2. **Correcting the positivity.** Before extending to the boundary as in [GKZ08], we need to correct the positivity of our regularizing function. The construction above creates a well behaved regularizing function on compact subsets. However, we require plurisubharmonicity to use the stability estimate.

As in [GGZ23b, Page 20], since X has an isolated singularity, we get the following argument: The exceptional divisor E of the resolution has the property that there exist positive rational numbers $(b_i)_{i\in I}$ such that $-\sum_{i\in I} b_iE_i$ is π-ample, where each E_i is an irreducible component of E. Then, on each $\mathcal{O}_{\widetilde{X}}(E_i)$ pick a section s_i cutting out E_i and choose an appropriate smooth Hermitian metric h_i such that

$$\rho' := B_1(\pi^*(B_0\rho) + \sum_{i\in I} b_i \log|s_i|^2_{h_i})$$

is strictly psh in $\widetilde{X}$ for some $B_0 > 1$ big enough and choose $B_1 > 0$ such that $dd^c\rho' \geqslant \tau$. Assume without loss of generality $\widetilde{u} \geqslant 0$. Now we fix:

[2]By [DDGHKZ14, Page 624] and [LPT21, Page 2036], taking β and τ not closed will change the exponencial map but not the estimates.

$$\widetilde{u}_\delta := \widetilde{u}_{c,\delta}; \quad Ac := H(\delta) - K\delta \tag{3.1}$$

for $H : \mathbb{R}^+ \to \mathbb{R}^+$ continuous, increasing function such that $\omega_H(\delta) \gtrsim \omega_v(\delta) \gtrsim O(\delta)$ and $H(0) = 0$, for v one of the barriers in Proposition 2.1 and A, K from Lemma 3.1. Take $\delta < \delta_2 \leqslant \delta_1$ for δ_2 small enough such that the c above is positive. We have control on its loss of positivity, hence:

$$\widehat{u}_\delta := \widetilde{u}_\delta + (Ac + K\delta)\rho' = \widetilde{u}_\delta + H(\delta)\rho'.$$

We have $dd^c\widehat{u}_\delta = dd^c\widetilde{u}_\delta + (Ac + K\delta)dd^c\rho'$. By the definition of ρ' and Lemma 3.1 we get $\widehat{u}_\delta \in PSH(\widetilde{\Omega}_\delta)$. Now, push down $\widehat{u}_\delta$ to X.

We define $\check{u}_\delta := \pi_*\widehat{u}_\delta$, which will be a singular psh function on Ω_δ.

3.3. **Extending regularization to all Ω.** Here, we will use the Hölder barriers constructed in Section 2 to control u $\delta-$close to the boundary. The extension will be, for some constant $C > 0$:

$$\overline{u}_\delta = \begin{cases} \max\{\check{u}_\delta, u + 4CH(\delta)\} & \text{in } \Omega_{2\delta}, \\ u + 4CH(\delta) & \text{in } \Omega \setminus \Omega_{2\delta}. \end{cases}$$

Notice that because $\check{u}_\delta$ goes to $-\infty$ at 0, then close to the singularity (inside some ball $B_\mu(0)$, for $\mu > 0$ small enough) $u + 4CH(\delta)$ will eventually be the greater term inside the max above, making $\overline{u}_\delta$ continuous and psh in Ω. For the gluing to be in $PSH(\Omega)$ we need that $\check{u}_\delta(x) \leqslant u(x) + 4CH(\delta), \forall x \in \partial\Omega_{2\delta}$. The barriers will give us this last inequality through the following arguments:

3.3.1. *Behaviour at the boundary.* Fix some $\lambda_0 > 0$ small enough such that $\overline{B_{\lambda_0}(0)} \Subset \Omega_{\delta_0}$.

Lemma 3.2. *The solution $u = u(\Omega, \phi, f)$ satisfies, for some constant $C' > 0$:*

$$|u(x) - u(\xi)| \leqslant C'\omega_v(d(x, \xi)), \forall x \in \Omega \setminus B_{\lambda_0}(0); \forall \xi \in \partial\Omega.$$

Proof. From Proposition 2.1:

$$v(x) - v(\xi) \leqslant u(x) - \phi(\xi) \leqslant -(w(x) - w(\xi)), \forall x \in \Omega \setminus B_{\lambda_0}(0); \forall \xi \in \partial\Omega.$$

By the modulus of continuity of the barriers we get, for some constant $C' > 0$:

$$|u(x) - u(\xi)| \leqslant C'\omega_v(d(x, \xi)), \forall x \in \Omega \setminus B_{\lambda_0}(0); \forall \xi \in \partial\Omega.$$

□

3.3.2. *Behaviour near the boundary.*

Lemma 3.3. *Take $r_0 > r > 0$ where $\overline{B_{r_0}(\xi)} \cap \Omega \subset \Omega \setminus B_{\lambda_0}(0); \forall \xi \in \partial\Omega$. For any $\xi \in \partial\Omega$, Then the solution u satisfies the following property:*

$$|u(x_1) - u(x_2)| \leqslant 2C'\omega_v(r)$$

for some constant $C' > 0$ and $\forall x_1, x_2 \in \overline{B_r(\xi)} \cap \Omega$.

Proof. Fix $r > 0$ and an arbitrary $\xi \in \partial\Omega$. Take any two points $x_1, x_2 \in \overline{B_r(\xi)} \cap \Omega$. Using the triangular inequality we get:

$$\begin{aligned} |u(x_1) - u(x_2)| &\leqslant |u(x_1) - u(\xi)| + |u(\xi) - u(x_2)| \\ &\leqslant C\omega_v(d(x_1, \xi)) + C\omega_v(d(\xi, x_2)) \\ &\leqslant 2C\omega_v(r). \end{aligned}$$

The second line comes from Lemma 3.2. □

3.3.3. *Extension of regularization.* By construction, the value of $u_\delta := \pi_* \widetilde{u}_\delta$ is under control relative to the supremum on a ball of radius δ for the points away from the singularity, where the resolution is an isomorphism. To prove the inequality necessary for the glueing process, we use an argument by contradiction:

Proof. Assume by contradiction that $\exists x_0 \in \partial\Omega_{2\delta}$ such that $\check{u}_\delta(x_0) > u(x_0) + 4CH(\delta)$. since $u_\delta(x_0) \leqslant \max\limits_{B_\delta(x_0)} u$, then $\exists\, x^* \in \overline{B_\delta(x_0)}$ such that $u(x^*) + H(\delta)\pi_*\rho'(x_0) > u(x_0) + 4CH(\delta)$. The points on $\partial\Omega_{2\delta}$ have distance 2δ to $\partial\Omega$, then take $\xi_0 \in \partial\Omega$ such that $d(\xi_0, x_0) = 2\delta$. We use the Lemma 3.3 for $r = 4\delta$, $x_1 = x_0$ and $x_2 = x^*$, for $\delta < \min\{r_0/2, \delta_2\}$. Then:

$$|u(x_0) - u(x^*)| \leqslant 2C'\omega_v(4\delta).$$

By assumption $|u(x^*) - u(x_0)| > 4CH(\delta) - H(\delta)\pi_*\rho'(x_0) \geqslant 8C'H(\delta)$. For $C \geqslant 2C' + \inf\limits_{\partial\Omega_\delta} \pi_*\rho' + \sup\limits_{\Omega} \pi_*\rho' > 0$. Then we get: $2C'(\omega_v(4\delta) - 4H(\delta)) > 0$, which is a contradiction for any $\omega_v(\delta) \gtrsim \delta$, as $\omega_H(\delta) \gtrsim \omega_v(\delta)$. □

Hence, the global extension $\overline{u}_\delta$ is continuous and psh as we wanted to construct.

4. L^1 Estimate

We will use the local theory developed in [GKZ08], and then refined in [BKPZ16, Cha17], to obtain the L^1 estimate of the regularizing function using a Laplacian estimate.

4.1. L^1 **Laplacian estimate.** Here, we calculate the estimate that dictates the behavior away from the boundary, derived from the local theory using a Laplacian estimate. First, we compare the local regularizing function with the one constructed in Section 3:

Lemma 4.1. *For $\varepsilon > 0$ small enough and $u \in PSH(\Omega) \cap C^0(\overline{\Omega})$ with $(dd^c u)^n = f\beta^n$ in Ω. Then for any sufficiently small $\delta > 0$ and chart on the resolution $(U \Subset \widetilde{\Omega_{\delta/2}}, \psi)$ we have:*

$$\eta_{\delta/2}\widetilde{u} \leqslant C_1 \Lambda_{\delta/2}\widetilde{u} + C_2\delta^2$$

for some constants $C_1, C_2 > 0$ and $\Lambda_\delta \widetilde{u}(\widetilde{x}) := \Lambda_\delta(\widetilde{u} \circ \psi^{-1})(\psi(\widetilde{x}))$ for any $\widetilde{x} \in U$.

Moreover, we have $||\eta_\delta \widetilde{u} - \widetilde{u}||_{L^1(\widetilde{\Omega_{2\delta}})} \leqslant \widetilde{C}\delta^{1-\varepsilon}$, for some constant $\widetilde{C} > 0$. Also, if $\Delta\widetilde{u}$ has finite mass in $\widetilde{\Omega}$ then one can get δ^2 instead of $\delta^{1-\varepsilon}$.

Proof. For a fixed chart $(U \Subset \widetilde{\Omega_{\delta/2}}, \psi)$ we have from the proof of [DDGHKZ14, Lemma 2.3]:

$$\eta_{\delta/2}\widetilde{u}(\widetilde{x}) = \frac{1}{(\delta/2)^{2n}} \int_{\widetilde{y} \in \widetilde{X}} \widetilde{u}(\widetilde{y}) \eta\left(\frac{|\log_{\widetilde{x}} \widetilde{y}|}{(\delta/2)^2}\right) dV_\tau(\log_{\widetilde{x}} \widetilde{y})$$

where $\widetilde{y} \mapsto \xi = \log_{\widetilde{x}} \widetilde{y}$ is the inverse function of $\xi \mapsto \widetilde{y} = \exp_{\widetilde{x}}(\xi)$. By the proof of [DDGHKZ14, Lemma 2.4] we have:

$$dV_\tau(\log_{\widetilde{x}} \widetilde{y}) = \bigwedge_{j=1}^{n} \frac{i}{2}(dz_j - dw_j) \wedge (d\bar{z}_j - d\bar{w}_j) + O(d_\tau(\widetilde{y}, \widetilde{x})^2),$$

where $(w, z) \mapsto (\widetilde{x}, \log_{\widetilde{x}} \widetilde{y}))$ represent local coordinates on a neighborhood of the zero section in TU, while $(\widetilde{x}, \log_{\widetilde{x}} \widetilde{y})) \mapsto (\widetilde{x}, \widetilde{y})$ is a diffeomorphism from that neighborhood onto the diagonal in $U \times U$. The $O(\cdot)$ term depends only on the curvature. By definition, coupled with the above equality:

$$\eta_{\delta/2}\widetilde{u}(\widetilde{y}) - \Lambda_{\delta/2}\widetilde{u}(\widetilde{y}) \leqslant C_1 \Lambda_{\delta/2}\widetilde{u}(\widetilde{y}) + C_2\delta^2,$$

for any $\widetilde{y} \in U$ and some constants $C_1, C_2 > 0$. Moreover, taking $\sup \eta = 1$ one can choose $C_1 = 0$, consequently applying Theorem 1.7, one gets:

$$||\eta_{\delta/2}\widetilde{u}-\widetilde{u}||_{L^1(U)}\leqslant C_3\delta^{1-\varepsilon},$$

for some constant $C_3>0$ for the chart (U,ψ). Then, by compactness of $\overline{\widetilde{\Omega_\delta}}$, the desired estimate is achieved for some constant $\widetilde{C}>0$:

$$||\eta_\delta\widetilde{u}-\widetilde{u}||_{L^1(\widetilde{\Omega_{2\delta}})}\leqslant \widetilde{C}\delta^{1-\varepsilon}.$$

□

Remark 4.2. *The proof doesn't involve the boundary values of u. This is clear by the proof of* [Cha17, Lemmas 3.2 and 3.3].

4.2. **Main estimate.** Theorem 4.3 bellow gives us our main estimate Theorem A and also Corollary B:

Theorem 4.3. *The unique solution $u=u(\Omega,\phi,f)\in PSH(\Omega)\cap C^0(\overline{\Omega})$ to $MA(\Omega,\phi,f)$, satisfies for any $x\in\Omega$:*

$$\omega_{u,x}(t)\leqslant C_x\omega_H(t)$$

for some constants $C_x>0$ such that $C_x\to+\infty$ as $x\to 0$, $0\leqslant\gamma<\frac{1}{(nq+1)}$ and $\omega_H(t)\gtrsim$ $\max\{\omega_\phi(t^{1/2}),t^\gamma\}$.

Proof. Fix $\gamma<\frac{1}{(nq+1)}$ and $\varepsilon>0$ sufficiently small that $\gamma':=\frac{\gamma}{(1-\varepsilon)}<\frac{1}{(nq+1)}$. Applying Theorem 1.6 for γ':

$$\begin{aligned}\sup_{x\in\Omega}(\overline{u}_\delta(x)-u(x)-4CH(\delta)) \leqslant&\sup_{x\in\partial\Omega}(\overline{u}_\delta(x)-u(x)-4CH(\delta))^*+\overline{C}||\overline{u}_\delta-u-4CH(\delta)||^{\gamma'}_{L^1(\Omega)}\\ \leqslant&\overline{C}||\overline{u}_\delta-u-4CH(\delta)||^{\gamma'}_{L^1(\Omega_{2\delta}\setminus B_\mu(0))}\\ \leqslant&\overline{C}||\check{u}_\delta-u-4CH(\delta)||^{\gamma'}_{L^1(\Omega_{2\delta}\setminus B_\mu(0))}\\ \leqslant&\overline{C}||\pi_*\widetilde{u}_\delta-u||^{\gamma'}_{L^1(\Omega_{2\delta}\setminus B_\mu(0))}\\ \leqslant&\overline{C}||\pi_*\eta_\delta\widetilde{u}-u||^{\gamma'}_{L^1(\Omega_{2\delta}\setminus B_\mu(0))}\\ \leqslant&\overline{C}||\eta_\delta\widetilde{u}-\widetilde{u}||^{\gamma'}_{L^1(\widetilde{\Omega_{2\delta}}\setminus\pi^{-1}[B_\mu(0)])}\\ \leqslant&\overline{C}(\widetilde{C}^{\gamma'}\delta^{(1-\varepsilon)\gamma'})\\ \leqslant&C_0\delta^\gamma\end{aligned}$$

Close to $\partial\Omega$ and 0 the terms are controlled by construction. From lines 3 to 4, we use the fact that $H(\delta)\rho'-4CH(\delta)\leqslant 0$. Remember that outside the singularity/divisors π is an isomorphism. The second to last passage is achieved by applying Lemma 4.1. Hence, we get:

$$\begin{aligned}C_0\delta^\gamma+4CH(\delta)\geqslant&\sup_{x\in\Omega}(\overline{u}_\delta(x)-u(x))\\ \geqslant&\overline{u}_\delta(x)-u(x),\quad\forall x\in\Omega\\ \geqslant&\pi_*\widetilde{u}_\delta(x)-u(x)+H(\delta)\pi_*\rho'(x),\quad\forall x\in\Omega_\delta\\ \geqslant&\widetilde{u}_\delta(\widetilde{x})-\widetilde{u}(\widetilde{x})+H(\delta)M\log(d_\tau(\widetilde{x},E)),\quad\forall\widetilde{x}\in\widetilde{\Omega_\delta}\end{aligned}$$

As $\rho'(\widetilde{x})\geqslant M\log(d_\tau(\widetilde{x},E))$, for any $\widetilde{x}\in\widetilde{\Omega}$ and some constant $M>0$. Then, for $S(\widetilde{\lambda}):=M(-\log(d_\tau(\widetilde{x},E)))$:[3]

[3]The notation $\widetilde{\lambda}(\widetilde{x})$ in $\widetilde{X}$ is the analogous of $\lambda(x)$ in X.

$$\widetilde{u}_\delta(\widetilde{x}) - \widetilde{u}(\widetilde{x}) \leqslant (C_1 + S(\widetilde{\lambda}))H(\delta), \quad \forall \widetilde{x} \in \widetilde{\Omega_\delta} \tag{4.1}$$

Following the proof of [DDGHKZ14, Theorem D], we get a uniform lower bound on the parameter $t = t(\widetilde{x})$ that realizes the infimum in the Kiselman-Legendre transform for $\widetilde{u}_\delta$ at a fixed $\widetilde{x} \in \widetilde{\Omega_\delta}$:

$$\widetilde{u}_\delta(\widetilde{x}) - \widetilde{u}(\widetilde{x}) = \eta_t \widetilde{u}(\widetilde{x}) + Kt^2 - \widetilde{u}(\widetilde{x}) - K\delta^2 - c\log(t/\delta) \leqslant (C_1 + S(\widetilde{\lambda}))H(\delta)$$

As $t \mapsto \eta_t \widetilde{u} + Kt^2$ is increasing, we have $\eta_t \widetilde{u} + Kt^2 - \widetilde{u} \geqslant 0$; thus:

$$c\log(t/\delta) \geqslant -(C_2 + S(\widetilde{\lambda}))H(\delta).$$

Since $c = A^{-1}H(\delta) - A^{-1}K\delta = H(\delta)(A^{-1} - A^{-1}K\frac{\delta}{H(\delta)})$, we get the bound $c \geqslant \frac{A^{-1}H(\delta)}{2}$, by choosing $\delta \leqslant \delta_3 := \min\{r_0/2, \delta_0, \delta_1, \delta_2, \varpi^{-1}(1/2K)\}$ for $\varpi(\delta) = \frac{\delta}{H(\delta)}$, because $\omega_H(\delta) \gtrsim \delta$. Then:

$$\delta \geqslant t(\widetilde{x}) \geqslant \delta\kappa(\widetilde{x}),$$

where

$$\kappa(\widetilde{x}) := e^{-2A(C_2+S(\widetilde{\lambda}))} = e^{-2AC_2'} \cdot (\widetilde{\lambda}(\widetilde{x}))^{2AM} \tag{4.2}$$

Finally, using that $t \mapsto \eta_t \widetilde{u} + Kt^2$ is increasing, $t(\widetilde{x}) \geqslant \delta\kappa(\widetilde{x})$ and the inequality (4.1), for every $\widetilde{x} \in \widetilde{\Omega_\delta}$ we get:

$$\eta_{\delta\kappa(\widetilde{x})}\widetilde{u}(\widetilde{x}) - \widetilde{u}(\widetilde{x}) - K\delta^2 \leqslant \widetilde{u}_\delta(\widetilde{x}) - \widetilde{u}(\widetilde{x}) \leqslant (C_1 + S(\widetilde{\lambda}))H(\delta) \tag{4.3}$$

which leaves us with:

$$\eta_\delta \widetilde{u}(\widetilde{x}) - \widetilde{u}(\widetilde{x}) \leqslant (C_3 + S(\widetilde{\lambda}))H(\delta/\kappa(\widetilde{\lambda})) \leqslant C_{\widetilde{\lambda}}\omega_H(\delta), \quad \forall \widetilde{x} \in \widetilde{\Omega_\delta} \tag{4.4}$$

by the subadditivity of ω_H we get: $C_{\widetilde{\lambda}} = \frac{(C_3 + S(\widetilde{\lambda}))}{\kappa(\widetilde{\lambda})} = \frac{[C_3 + M'(-\log(\widetilde{\lambda}(\widetilde{x})))]}{(\widetilde{\lambda}(\widetilde{x}))^{2AM}}$.

Fix $\widetilde{x} \in \widetilde{\Omega} \setminus E$, then there exists $\delta' \leqslant min\{1/2, \delta_3, \lambda(x)/2, d(x, \partial\Omega)/2\}$. Now fix any $0 < \delta < \delta'$, hence $\widetilde{x} \in \widetilde{\Omega_\delta}$. Applying [Ze20, Theorem 3.4] for $\overline{\widetilde{\Omega_\delta}}$, $\widetilde{u}$ and inequality (4.4):

$$|\widetilde{u}(\widetilde{x}) - \widetilde{u}(\widetilde{y})| \leqslant D_0(C_{\widetilde{\lambda}(\widetilde{x})} + C_{\widetilde{\lambda}(\widetilde{y})})\omega_H(d_\tau(\widetilde{x}, \widetilde{y})), \tag{4.5}$$

for any $\widetilde{y} \in \widetilde{\Omega_\delta}$ and some constant $D_0 > 0$. Then, one can pass the right-hand side of (4.5) to Ω_δ and substitute $\widetilde{\lambda}$ by λ, by choosing $\tau := \widetilde{C}\pi^*\beta + \varepsilon\theta$ for some constants $\widetilde{C} > 1$ big enough and $0 < \varepsilon$ small enough, with some smooth closed (1,1)-form θ so that $\pi^*d \leqslant d_\tau$. Hence:

$$|u(x) - u(y)| \leqslant (C_{\lambda(x)} + C_{\lambda(y)})\omega_H(d_\tau(\widetilde{x}, \widetilde{y})), \tag{4.6}$$

for any $\widetilde{y} \in \widetilde{\Omega_\delta}$ and some constant $C_\lambda > 0$ that goes to $+\infty$ as $\lambda \to 0$. Lastly, notice that we have the comparison $\tau_{\widetilde{x}} \leqslant K_{\lambda(x)}\pi^*\beta_{\widetilde{x}}$ for some constant $K_\lambda > 0$ that goes to $+\infty$ as $\lambda \to 0$. Now, by the subadditivity of ω_H we get:

$$|u(x) - u(y)| \leqslant (C'_{\lambda(x)})\omega_H(d(x, y)) \leqslant (C'_{\lambda(x)})\omega_H(\delta) \tag{4.7}$$

for any $y \in B_\delta(x)$ and some constant $C'_\lambda > 0$ that goes to $+\infty$ as $\lambda \to 0$. By choosing $\omega_H(t) = \max\{\omega_\phi(t^{1/2}), t^\gamma\}$ one proves Theorem A, Corollary B follows as $\omega_\phi(t) = t^\alpha$. □

Remark 4.4. *One can notice that the proof of* [Ze20, Theorem 3.4] *follows even if* $\overline{\widetilde{\Omega_\delta}}$ *have a boundary because* $\widetilde{u}$ *is defined on all* $\widetilde{\Omega}$.

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