THE HERMITIAN CALABI-YAU THEOREM

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Abstract. This master thesis has as its main goal to provide a pluripotential-theoretic proof of the Hermitian version of the Calabi-Yau Theorem (HCY for short), see Theorem 2.7 for the statement. This approach differs from the one on the first proof of the theorem given by Tossati-Weinkove [TW10a, TW10b] which follows more the spirit of Yau's proof for the Calabi conjecture [Yau78]. We also discuss a few of the dificulties and progresses on the use of pluripotential theory for the Hermitian setting.

Contents

1. Introduction

In Differential Geometry constructing canonical metrics is a central problem. In the complex context we have remarkable results such as Yau's proof [Yau78] to the Calabi conjecture [Cal57] and the Chen-Donaldson-Sun proof [CDS15] to the Yau-Tian-Donalson conjecture to cite ones related to Kähler-Einstein metrics. However, both results cited before are on the Kähler setting. For Hermitian manifolds a natural generalization of the Calabi Conjecture (see Theorem below or the re-statement of it in 2.7)

Theorem 1.1 ([TW10b]). *Let (X, ω) be a compact Hermitian manifold. Then every representative of the first Bott-Chern class $c_1^{BC}(X)$ can be represented as the Ricci form of a metric of the type $\omega + i\partial\bar{\partial}\varphi$ for some $\varphi \in C^\infty(X, \mathbb{R})$.*

remained open for 30 years after Yau's proof on the Kähler case. Providing a Pluripotential Theoretic proof of the theorem above is the main goal of this master thesis.

Even though the effort of many mathematicians ([Gau84, Chr87, Ha96, GuaLi10, TW10a] to site a few) only partial solutions where given such as solutions with metrics with special conditions. [1] In 2010 Tossati-Weinkove [TW10b] completed the proof using techniques inspired by [Yau78].

To solve such problems it is common to reduce it to solving a *Complex Monge-Ampère Equation* (in our case the Monge-Ampère (MA) Problem bellow or its re-statement in Theorem 2.8)

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[1] Conditions such as Balanced (see [TW10a, Din19]) and the Guan-Li [GuaLi10] condition see Definition 2.2.

Theorem 1.2. *Let (X, ω) be a compact Hermitian manifold of complex dimension n. Let also f be any smooth strictly positive function on X. Then the following problem*

$$\begin{cases} u \in \mathcal{C}^\infty(X), \quad \omega + dd^c u > 0, \\ \sup_X u = 0, \\ c \in \mathbb{R}, \\ (\omega + dd^c u)^n = e^c f \omega^n, \quad f \in \mathcal{C}^\infty(X), \quad f > 0. \end{cases}$$

admits a unique solution (u, c). Furthermore there exist constants $C_k, k = 0, 1, 2, \ldots$ dependent only on X, ω and f, such that the C^k-th norm of the function u is bounded by C_k.

Solving this MA Problem above is equivalent to solving the first theorem (Hermitian Calabi-Yau (HCY) Theorem). And this is the approach that we will use on this text to present a proof of the HCY Theorem.

Many different techniques are needed to attack such problems (see the outline of the proof between Definition 2.6 and Theorem 2.7), one very important is the *Moser Iteration Technique* used in [Yau78, TW10a, TW10b]. However, another very effective one is Pluripotential Theory. Inspired by the works of Bedford-Taylor [BT76, BT82] this theory has been extended to the Kähler and Hermitian setting.

To cite some on the Hermitian setting of the MA Problem we have for L^p densities f, Dinew–Kołodziej [DK12] used pluripotential techniques to obtain uniform L^∞-estimates. Then, regarding solvability Kołodziej–Nguyen [KN15] proved it using a stability estimate. Also, Guedj–Lu [GL21] established uniform estimates and proved the existence of solution when the (1, 1)-form ω is merely big. As a consequence, they generalized Tosatti–Weinkove's theorem 2.7 to a singular Hermitian setting, more precisely to Hermitian $\mathbb{Q}$-Calabi–Yau varieties.

The reader can also find a few more words regarding other problems on the Appendix A. Like the generalization of the HCY Theorem known as the Calabi-Yau-Gauduchon Theorem established in [STW17] and a few coments on difficulties and developments of Puripotential Theory to attack the Singular setting.

1.1. **Structure of text.** This text consist mostly of a proof of the HCY Theorem. In Section 2 we introduce some necessary language and start the proof and make the link between HCY and the MA Problem (link between Theorem 2.7 and Theorem 2.8). Sections 3 4 5 6 are the technical part of the proof which consists of the many steps to prove Theorem 2.8. The Appendix A we briefly mention some problem one can pursue the understanding after reading this text, namely the Calabi-Yau-Gauduchon Theorem and the problem of extending Pluripotential Techniques to the Singular setting. The Appendix B is a list of Lemmas used during the text and references where to find their proofs.

2. From HCY to Monge-Ampère

This section starts with a few definitions for context and has the statement and proof of Theorem 2.7. This proof consists on transforming the original geometrical problem in solving the *Complex Monge-Ampère* (Theorem 2.8) and then solving this equation. To be able to do that we'll assume the results on the following sections. With this we can conclude this less technical part and then understand these result that we use. The results on the following sections are the real difficulty in solving such an equation. Here the continuity method is explained and used to solve the Monge-Ampère. Our approach here comes from [Din19, GZ17, Blo12], the last two for the Kähler setting.

For the whole text we fix (X, ω) a $n-$*dimensional* Hermitian manifold with *Hermitian form* ω. Remember that the Hermitian form, a real (1,1)-form can be described as:

$$\omega = i \sum_{j,k} g_{j\bar{k}} dz^j \wedge \overline{dz^k}$$

where the coefficients $g_{j\bar{k}}$ are smooth local complex valued functions, such that pointwise $g_{j\bar{k}}(z)$ is a positive definite Hermitian symmetric matrix.

It should be noted that ω does not need to be closed. If it is, then we are on the Kähler case.

There are many special types of Hermitian metrics, besides Kähler, that are less restrictive topologically. Here are some that will appear throughout this text.

Definition 2.1. A hermitian metric is said to be *Gauduchon* if the hermitian form ω has the property $dd^c(\omega^{n-1}) = 0$.

By a result of Gauduchon [Gau77] every Hermitian manifold admits a Gauduchon metric and given a metric ω you can always find a Gauduchon function ϕ such that $e^{\phi}\omega$ is Gauduchon.

Now we define a metric condition studied by Guedj-Lu [GL22] in the singular case called condition (B). This condition was studied by Guan-Li [GuaLi10] when $B = 0$.

Definition 2.2. (X, ω) is said to have the *condition* (B) if exists positive B such that:

$$-B\omega^2 \leqslant dd^c\omega \leqslant B\omega^2 \text{ and } -B\omega^3 \leqslant d\omega \wedge d^c\omega \leqslant B\omega^3$$

By compacity and smoothness of our case of interest this constant always exist. A metric is called Guan-Li if $B = 0$ and it simplifies a bunch of calculations on Hermitian Geometry. More precisely, it is easy to see that if metric is Guan-Li the volume of the Monge-Ampère of a dd^c perturbation is constant. Moreover, due to Chiose [Chi16] it is known that Guan-Li condition is actually equivalent to having constant Monge-Ampère volume under dd^c perturbation.

To talk about pluripotential theory it is necessary to understand a few notions:

Definition 2.3. A function $u : \Omega \subset \mathbb{C}^n \to [-\infty + \infty[$ is plurisubharmonic (pluriharmonic) if it is upper semi-continuous and for all complex lines $\Lambda \subset \mathbb{C}^n$, the restriction $u \mid_{\Omega\cap\Lambda}$ is subharmonic (harmonic) in $\Omega \cap \Lambda$.

The latter property can be reformulated as follows: for all $a \in \Omega, \xi \in \mathbb{C}^n$ with $|\xi| = 1$, and $r > 0$ such that $\bar{B}(a, r) \subset \Omega$,

$$u(a) \leqslant \frac{1}{2\pi}\int_0^{2\pi} u\left(a + re^{i\theta}\xi\right) d\theta$$

Definition 2.4. The ω-plurisubharmonic functions are the elements of the function class

$$PSH_\omega(X) := \left\{u \in \mathcal{C}^\uparrow(X) \cap L^1(X, \omega) : dd^c u \geqslant -\omega\right\}$$

where $\mathcal{C}^\uparrow(X)$ denotes the space of upper semicontinuous functions and the inequality is understood in the weak sense of currents.

There plurisubharmonic (psh) functions are the functions for the local Pluripotential Theory. The ω-plurisubharmonic (ω-psh) are their global counterparts as for compact manifolds all psh functions are constant.

Related to our problem and Pluripotential Theory in Complex Geometry the Monge-Ampère measure is very central. The theory to define this types of operators in the context of bounded functions (on the local case) date back to works of Bedford-Taylor, see [BT76, BT82]. For our purposes we define it as: consider u a bounded ω-psh function on X. Take $U \subset X$ open which is biholomorphic to the unit ball in $\mathbb{C}^n$, and fix ρ smooth strictly[2] psh function in U such that $dd^c\rho \geqslant \omega$. Let, $\varphi := u + \rho$ and we define:

$$(\omega + dd^c u)^n := (\omega - dd^c\rho + dd^c\varphi)^n = \sum_{k=0}^{n}\binom{n}{k}(\omega - dd^c\rho)^k \wedge (dd^c\varphi)^{n-k}.$$

We finally get that although the $(1, 1)$-form $\omega - dd^c\rho$ is not semipositive it is smooth and by Bedford-Taylor Theory the (signed) measures $(\omega - dd^c\rho)^k \wedge (dd^c\varphi)^{n-k}$ are well defined. And so are the *mixed Monge-Ampère measures* $(\omega + dd^c u)^j \wedge (\omega + dd^c v)^{n-j}$ for u, v bounded ω-psh functions.

Another important definition which is crucial for our study is the *Bott-Chern Cohomology* defined just bellow. It will be the appropriate Cohomolgy for our problem and substitute the

[2] Here it mean that the inequality on the definition is strict.

classical DeRham Cohomology used on the Kähler case. They are equivalent on the Kähler setting because on this case one has the important $\partial\bar{\partial}$*Lemma* that dates back to the work of Hodge (see [GZ17] for a few words on it).

Definition 2.5. We define the p, q *Bott* $-$ *Chern Cohomology Group* as:

$$H^{p,q}_{BC} = \frac{\ker\{d : \Omega_{p,q}(X) \to \Omega_{p,q+1}(X) \oplus \Omega_{p+1,q}(X)\}}{\operatorname{Im}\{dd^c\Omega_{p-1,q-1}(X)\}}$$

where $\Omega_{p,q}(X)$ denotes the space of smooth (p, q)-forms.

Definition 2.6. Given a Hermitian metric ω its *Ricci form* can be defined analogously to the Kähler setting by

$$\operatorname{Ric}(\omega) := -dd^c \log(\omega^n).$$

here we use the convention of $d = \partial + \bar{\partial}$ and $d^c = \frac{1}{2i}(\partial - \bar{\partial})$, such that $dd^c = i\partial\bar{\partial}$.

The last ingredient we need to enunciate the main theorem is the *first Bott-Chern cohomology class* $c_1^{BC}(X)$ in the Bott-Chern Cohomology that can be defined anagolously as in the Kähler case (for more details of the Kähler case [GZ17]). For our matters the reader unfamiliar should just keep in mind that it is an element of H^2_{BC} and an invariant.

Now we will present an outline of the long proof of the main theorem and the techniques on each step:

- Going from the HCY to a MA Problem of the Monge-Ampère Equation. Section 2. *(Geometrical calculations.)*
- Proving Uniqueness. Section 3. *(Laplace-Beltrami Operator Theory and Pluripotential Theory.)*
- Using Continuity Method to reduce the problem to proving openess and closedness of the parameter set for the solutions of the continuity method. Section 2.
- Proving openness of the parameters set. Section 3. *(Laplace-Beltrami Operator Theory and Inverse Function Theorem)*
- Proving Closedness of the parameters set. Which can be devided as:
  - Proving an $L^\infty -$ *estimate*. Section 4. *(Pluripotential Theory)*
  - Proving an $\Delta -$ *estimate*. Section 5. *(Geometrical Inequalities)*
  - Proving an $C^{2,\alpha} -$ *estimate*. Section 6. *(Complex Evans-Krylov Theory)*
  - Proving an $C^{2+k,\alpha} -$ *estimate*. Section 6. *(Schauder Theory)*
  - Use a Arzelà-Ascoli argument, using the a priori estimates to finish proving the closedness. Section 2

Now we re-state the main theorem. And will use the results from the following sections to argue this less technical part of the proof, and then dive in the technicalities on the sections to come.

Theorem 2.7 ([TW10b])**.** *Let (X, ω) be a compact Hermitian manifold. Then every representative of the first Bott-Chern class $c_1^{BC}(X)$ can be represented as the Ricci form of a metric of the type $\omega + i\partial\bar{\partial}\varphi$ for some $\varphi \in C^\infty(X, \mathbb{R})$.*

Proof. Fix $\eta \in c_1^{BC}(X)$. Analogously as on the Kähler case we have (because the definition of the Bott-Chern Cohomology, from the point of view of computations, encapsulates the $\partial\bar{\partial}$*Lemma*) that the Ricci form is inside the first chern class also in this case.[3] Then, by the definition of our cohomology:

There exists $h \in \mathcal{C}^\infty(X, \mathbb{R})$ such that

$$\operatorname{Ric}(\omega) = \eta + dd^c h$$

[3]Actually there should be a multiplicative positive constant to make in exactly but such constant does not affect the calculations in a meaning full way so it is usual, for convinience, to just ignore it.

We search for $\omega_\varphi := \omega + dd^c\varphi$ a Hermitian form that comes from this $dd^c - pertubation$ such that $\operatorname{Ric}(\omega_\varphi) = \eta$. Since

$$\operatorname{Ric}(\omega_\varphi) = \operatorname{Ric}(\omega) - dd^c \log\left(\frac{(\omega + dd^c\varphi)^n}{\omega^n}\right),$$

Putting together the expressions for $Ric(\omega)$ and $Ric(\omega_\varphi)$ we get:

$$dd^c\left\{h - \log\left(\frac{(\omega + dd^c\varphi)^n}{\omega^n}\right)\right\} = 0.$$

Then the function inside the brackets is pluriharmonic, hence constant since X is compact. And with that we arrive at the equation:

$$(\omega + dd^c\varphi)^n = e^{h+c}\omega^n \tag{2.1}$$

Note that differently from the Kähler case here we can't really get rid of this constant c because the volume of the Monge-Ampère of the perturbation by dd^c is not fixed, since ω is not closed. [4] Because of that we also don't have the necessary condition of the normalization of the volume as one has on the Kähler setting.

Solving the Equation 2.1 is equivalent of solving the MA problem on Theorem 2.8. Then, assuming this theorem our proof is done.

□

We now state the MA problem as Theorem 2.8 and solve it assuming results from the next sections:

Theorem 2.8. *Let (X, ω) be a compact Hermitian manifold of complex dimension n. Let also f be any smooth strictly positive function on X. Then the following problem*

$$\begin{cases} u \in \mathcal{C}^\infty(X), \quad \omega + dd^c u > 0, \\ \sup_X u = 0, \\ c \in \mathbb{R}, \\ (\omega + dd^c u)^n = e^c e^h \omega^n, \quad h \in \mathcal{C}^\infty(X). \end{cases}$$

admits a unique solution (u, c). Furthermore there exist constants $C_k, k = 0, 1, 2, \dots$ dependent only on X, ω and f, such that the C^k-th norm of the function u is bounded by C_k.

Proof. To prove this theorem will use the Continuity Method and assume the following chapter to complete the proof. Using the Theorems 3.1, 3.3, 4.7, 5.2, 6.1, 6.2.

The *continuity method* consists in deforming our complicated PDE into one that is trivial. Then one tries to propagate the fact that we can solve the trivial one to get a solution on the original. More precisely we consider the family of problems:

$$(*)_t \quad \begin{cases} u_t \in PSH_\omega(X), \\ \sup_X u_t = 0, \\ (\omega + dd^c u_t)^n = e^{c_t} e^{th}\omega^n, \quad h \in \mathcal{C}^\infty(X). \end{cases}$$

for $t \in [0, 1]$. Clearly the problem $(*)_0$ is solvable, it is our trivial equation. $(*)_1$ is the MA Equation we want to solve. To propagate the solution we have to prove that the parameter set of the solutions of continuity method[5]

$$S := \{T \in [0, 1] \mid (*)_t \text{ is solvable for every } t \leqslant T\}$$

is open and closed in $[0, 1]$, because we already know it is non-empty and this will imply $S = [0, 1]$ which implies $(*)_1$ has a solution.

[4] Moreover, having this volume not fixed is a barrier for the extension of pluripotential techniques to the hermitian setting. More on it in Section A.

[5] Quite naturally there are many deformations to be chosen. For example in [TW10a] Tossati-Weinkove choose a different one but the proof can be done completely analogous at all steps.

Assuming we know the functions of the solutions of $(*)_t$ $\{u_t : t \in S\}$ are bounded in $C^{k+2,\alpha}(M)$ and that c_t and c_t^{-1} are bounded.[6] Which is what we get from the next few chapters where we build *a priori estimates* of the solutions. Then from every sequence, by the Arzela-Ascoli theorem, we can choose a subsequence whose all partial derivatives of order $\leqslant k+1$ converged uniformly and the sequence of constants c_t also converge uniformly. Then we get sequentially closed, which implies S is closed.

The last two parts (openess and uniqueness we'll see on the next section). The Openess we get from Theorem 3.3. The Uniqueness we get from Theorem 3.1. This concludes the proof of Theorem 2.8

□

3. Uniqueness and Openess

In this section we begin the technical details of this proof. Here uniqueness (Theorem 3.1) of the MA Problem of the Complex Monge-Ampère Equation in Theorem 2.8 is proven. Also the Openess (Theorem 3.3) of the parameter set for the solutions of the Continuity Method. Mostly here is used knowledge of Second Order Elliptic PDEs (as the Monge-Ampère is an example) and, by choice of the author, some pluripotential perspectives on the uniqueness. The openess comes from a perturbation argument using Inverse Function Theorem on Banach Manifolds. Our approach here comes from [Din19].

3.1. **Uniqueness.** In [TW10a] the authors proved that if u, v are smooth ω-psh functions and their Monge-Ampère measures satisfy $\omega_u^n = e^{c_1} f\omega^n, \omega_v^n = e^{c_2} f\omega^n$ for some smooth positive function f and some constants c_1 and c_2. Then we get $c_1 = c_2$ and $u - v$ is constant. This is will give us the uniqueness of solution on Theorem 2.8, because of normalization. This will imply unicity in the potential for the dd^c perturbation on the HCY.

Theorem 3.1. *If $(u, c_1), (v, c_2)$ are solutions to the MA problem 2.8. Then $c_1 = c_2$ and $u - v = 0$.*

Proof. If we assume $c_1 = c_2$ we have that $u = v$ is simple. Suppose that we already knew that $c_1 = c_2$. Then we have:

$$0 = e^{c_1} f\omega^n - e^{c_1} f\omega^n = \omega_u^n - \omega_v^n = dd^c(u - v) \wedge \left(\sum_{k=0}^{n-1} \omega_u^k \wedge \omega_v^{n-1-k}\right)$$

This can be treated as a linear strictly elliptic equation with respect to $u - v$. The coefficients of the form $\sum_{k=0}^{n-1} \omega_u^k \wedge \omega_v^{n-1-k}$ pointwise give strictly positive definite matrix. With that we can apply the strong maximum principle (see [GT83]) yields that $u - v$ must be a constant.

Now we show that $c_1 = c_2$. The proof can use a maximum principle argument due to Cherrier [Chr87]. However we'll use a more pluripotential theorectic approach due to Dinew-Kołodziej [DK12]. Suppose by contradiction:

$$\omega_u^n = e^{c_1} f\omega^n, \omega_v^n = e^{c_2} f\omega^n$$

for some smooth u, v and $c_1 \neq c_2$. We can without loss of generality assume that $c_2 > c_1$.

Consider the Hermitian metric $\omega + dd^c u$. This metric is an actual Hermitian metric (smooth, strictly positive) by the assumptions on the MA problem. Then by the Theorem of Gauduchon [Gau77] one can take a unique (up to aditive constant) function ϕ_u (*Gauduchon function*) that after a conformal transformation by it our new metric is Gauduchon (see Definition 2.1). Then we can have:

$$inf_X \phi_u = 0, dd^c\left(e^{(n-1)\phi_u}(\omega + dd^c u)^{n-1}\right) = 0.$$

Then one can apply the Comparison Principle for the Laplacian of the Gauduchon metrics (Proposition 4.1) on $e^{\phi_u}(\omega + dd^c u)$ which gives us:

[6] Bounds on the constant c_t are given by L^∞ estimate of Theorem 4.7.

$$\int_{\{u<v\}} e^{(n-1)\phi_u} (\omega + dd^c u)^{n-1} \wedge \omega_v \leqslant \int_{\{u<v\}} e^{(n-1)\phi_u} \omega_u^n.$$

We can with the same reasoning take v as big as we want by adding a constant. Then we obtain:

$$\int_X e^{(n-1)\phi_u} (\omega + dd^c u)^{n-1} \wedge \omega_v \leqslant \int_X e^{(n-1)\phi_u} \omega_u^n$$

With this inequality we can analyse the integrand on the left hand side pointwise to reduce estimating it to a linear algebra problem. With this we can co-reduce the matrixes for both metrics. This will reduce to a eigenvalue related calculation which can be estimated using AM-GM Inequality. Finaly arriving at:

$$\int_X e^{(n-1)\phi_u} (\omega + dd^c u)^{n-1} \wedge \omega_v \geqslant \int_X e^{(n-1)\phi_u + \frac{(c_2-c_1)}{n}} \omega_u^n$$

Puting both estimates together we obtain:

$$1 < e^{\frac{(c_2-c_1)}{n}} \leqslant 1$$

which contradicts our assumption.

□

3.2. **Openness.** The openness part boils down to showing that if $(*)_T$ is solvable then the problem $(*)_t$ is also solvable for t close enough to T. This is achieved by a Perturbation argument applying the implicit function theorem between well chosen Banach spaces and linearization of the equation. Here the linearized operator is essentially the Laplacian, and we shall prove that this operator is bijective in our setting. The details are taken from [TW10a] and [Din19].

Before stating the Openess precisely and proving it we'll analyse the solvability of the Laplacian in our Hermitian context with the following proposition:

Proposition 3.2. *Let ω be a Gauduchon metric on X and let Δ_ω be the Laplacian operator with respect to ω. Then, given any $f \in L^2(X,\omega)$ there is a unique $W^{2,2}$ function u which solves the problem*

$$\Delta_\omega u = f, \quad \int_X v\omega^n = 0$$

if and only if $\int_X f\omega^n = 0$. Furthermore if $\alpha \in (0,1)$ and $f \in \mathcal{C}^\alpha(X)$, then $u \in \mathcal{C}^{2,\alpha}(X)$

Proof. The proof of normalized solutions follows from the ellipticity of Δ_ω. By the following formal computation we'll have the adjoint of our operator.

$$\begin{aligned}\int_X < \Delta_\omega u, g > \omega^n &= \int_X g dd^c u \wedge \omega^{n-1} = \int_X u dd^c \left(g\omega^{n-1}\right) \\ &= \int_X \left(u dd^c g \wedge \omega^{n-1} + u dg \wedge d^c \left(\omega^{n-1}\right) - u d^c g \wedge d\left(\omega^{n-1}\right)\right) \\ &= \int_X < u, \Delta_\omega^* g > \omega^n\end{aligned}$$

Then the adjoint operator Δ_ω^* is second order elliptic. Moreover it contains no zero order term[7] thus it contains only constant functions in its kernel. On the other hand, again by classical elliptic theory the image of Δ_ω in L^2 is perpendicular to the kernel of Δ_ω^* which proves the first assertion. The second assertion is a consequence of the classical Schauder theory of linear elliptic equations.

□

[7]The Gauduchon metric condition actually is used here.

Now we can prove the Openess. For the Käher case the argument is analogous (see [Yau78],[Blo12]), this case there are only a few extra terms, but also the reader should keep in mind that the solutions here are of the form (u,c) and not only a function as in the Kähler case.

Theorem 3.3. *The parameters set of solutions of the continuity method defined on the proof of Theorem 2.8 is open in [0,1].*

Proof. Suppose $T \in S$, i.e; there is a smooth solution (u, c_T) to the problem $(*)_T$. Let ϕ_u denote the Gauduchon function, like in the proof of uniqueness above, associated to ω_u. We normalize this time differently to better suit our calculations. Choose it such that: $\int_X e^{(n-1)\phi_u}(\omega + dd^c u)^n = 1$. Fix a small positive constant $\alpha < 1$. This constant will be important during Sections 6 when we analyse Higher Order Estimates, it will also be relevant that it only depends on X, ω and n.

Consider now these two Banach manifolds:

$$B_1 := \left\{ w \in \mathcal{C}^{2,\alpha}(X) \mid \int_X w e^{(n-1)\phi_u}\omega_u^n = 0 \right\}$$

and

$$B_2 := \left\{ h \in \mathcal{C}^{\alpha}(X) \mid \int_X e^{h+(n-1)\phi_u}\omega_u^n = 1 \right\}$$

To construct the Perturbation Argument we will use a operator that takes values only on functions and solve the problem finding a function that satisfies our construction. However, it should be noted that by the structure of the problem in it self as soon as we find this fuction the constant that is it's companion to form a solution (u,c) will be determined. We'll now consider a operator to solve a perturbation problem as a framework and after we will dive in more details in the particular perturbation we are interested and why it solves what we need.

Consider our operator for the perturbation to be $\mathcal{T} : B_1 \to B_2$ given by

$$\mathcal{T}(v) := \log \frac{(\omega + dd^c u + dd^c v)^n}{(\omega + dd^c u)^n} - \log \int_X e^{(n-1)\phi_u}(\omega + dd^c u + dd^c v)^n$$

Note that $\mathcal{T}(0) = 0$ and that any function v sufficiently close to 0 in $\mathcal{C}^{2,\alpha}$-norm is $(\omega + dd^c u)-$plurisubharmonic.

By the Implicit Function Theorem for Banach Manifolds the equation $\mathcal{T}(v) = f$ is solvable for any $h \in B_2$ sufficiently close in $\mathcal{C}^{\alpha}$ norm to zero if the Frechet derivative

$$(D\mathcal{T}) : T_0 B_1 = B_1 \to T_0 B_2 = \left\{ g \in \mathcal{C}^{\alpha}(X) \mid \int_X g e^{(n-1)\phi_u}\omega_u^n = 0 \right\}$$

is an invertible linear mapping.

To prove it is invertible linear map we have:

$$(D\mathcal{T})(\eta) = \Delta_{\omega + dd^c u}\eta - n\int_X e^{(n-1)\phi_u}\omega_u^{n-1} \wedge dd^c \eta$$

By sucessive applications of Stokes Theorem (following the logic of Lemma B.5) and the Gauduchon condition we have that the second term is zero. The question is thus whether $\Delta_{\omega+dd^c u} : B_1 \to T_0 B_2$ is a continuous bijective mapping.

By Proposition 3.2 (this will justify the Gauduchon transformation we did on the original metric) the equation

$$\Delta_{e^{\phi_u}(\omega + dd^c u)}(\eta) = \tau$$

is solvable if and only if $\int_X \tau e^{n\phi_u}(\omega + dd^c u)^n = 0$ and the solution is unique up to an additive constant. Thus we can assume that $\int_X \eta e^{(n-1)\phi_u}\omega_u^n = 0$. Furthermore, if $\tau \in \mathcal{C}^{\alpha}(X)$ then η belongs to $\mathcal{C}^{2,\alpha}(X)$ and hence it belongs to B_1. Note that $\Delta_{e^{\phi_u}(\omega+dd^c u)}(\eta) = e^{-\phi_u}\Delta_{(\omega+dd^c u)}(\eta)$ thus $(D\mathcal{T})(\eta) = \tau$ is solvable if and only if $\int_X \tau e^{(n-1)\phi_u}(\omega + dd^c u)^n = 0$ i.e. exactly if τ belongs to $T_0 B_2$. This proves the surjectivity of $(D\mathcal{T})$ and injectivity follows from the normalization condition. Finally continuity of $(D\mathcal{T})$ follows from the Schauder $\mathcal{C}^{2,\alpha}$ a priori estimates for the Laplace equation.

Now we go back to the Perturbation we wanted to study: (in an interval $s \in [T, T+\varepsilon), \varepsilon > 0$)

$$(\omega + dd^c u + dd^c v)^n = e^{(s)h} e^{c_{T+s} - c_T} (\omega + dd^c u)^n$$

Here what we actually want is a continous curve of solutions $\alpha : [0, \varepsilon) \ni s \mapsto (u_s, c_{T+s})$ where $u_0 = u$ and which goes monotonically to the solution at zero as $s \to 0^+$.

By construction $sh, c_{T+s} - c_T$ both go to zero monotonously as s goes to zero.

Note that the equation we are solving with $\mathcal{T}(v) = f$ is:

$$(\omega + dd^c u + dd^c v)^n = (\int_X e^{(n-1)\phi_u} (\omega + dd^c u + dd^c v)) e^f (\omega + dd^c u)^n$$

Then it exists ε small enough such that $sh + c_{T+s} - c_T$ is close enough to zero for us to have a solution by the Inverse Function Theorem argument before. And consequentely, by the monotonicity, any $s \in [T, T+\varepsilon)$, for the ε above, will also admit a solution.

Then we get that for some v close enough to zero (which makes it $(\omega + dd^c u)$-plurisubharmonic) we get, because by hypothesis for T we have the solution:

$$(\int_X e^{(n-1)\phi_u} (\omega + dd^c u + dd^c v)) e^{sh + c_{T+s} - c_T} (\omega + dd^c u)^n = e^{\log(\int_X e^{(n-1)\phi_u} (\omega + dd^c u + dd^c v))} e^{(T+s)h + c_{T+s}} \omega^n$$

Then we have our solutions (u_{T+s}, c_{T+s}) by changing the constant c_{T+s} above by it minus the constant $\log(\int_X e^{(n-1)\phi_u} (\omega + dd^c u + dd^c v))$. [8] Hence, we get the openness wanted as we get that for any T which is has a smooth solution, because there always exists small enough ε such that $\forall s \in [T, T+\varepsilon)$, $(*)_s$ also admits a smooth solution.

□

4. L^∞ Estimate

Here we do the most challenging part of the proof, the L^∞ − *estimate*. Here we will use techniques of Pluripotential theory due to Guedj-Lu in [GL21]. For a alternative Pluripotential proof good references are [Din19, KN15, DK12]. Yau's original proof to the Calabi Conjecture used the Moser Iteration Technique, for surveys on this technique on the Kähler case see [Siu87] and [Blo12], for a proof in this fashion for the Hermitian case see [TW10a, TW10b]. This Section will contain brief informations on Pluripotential Theory. The clever argument on [Lu21] bypasses a lot of dificulties particular to doing Pluripotential Theory in the Hermitian setting. Our approach here comes from [Lu21, GL21].

Before starting with the Pluripotential proof of this section we enunciate (see [Din19] for a proof) the Comparison Principle for the Laplacian of the Gauduchon metrics. (used in the proof of Theorem 3.1.)

Proposition 4.1. *Let ω be a Gauduchon metric and let $\phi, \psi \in PSH_\omega(X) \cap L^\infty(X)$. Then*

$$\int_{\{\phi < \psi\}} \omega_\psi \wedge \omega^{n-1} \leqslant \int_{\{\phi < \psi\}} \omega_\phi \wedge \omega^{n-1}.$$

Now we prove a couple of important result in pluripotential theory before we enunciate and prove the L^∞ estimate.

We start with the following maximum principle.

Lemma 4.2. *Assume u, v are bounded w-psh functions on $U \subset X$. Fix smooth real $(1,1)$-forms $\omega_1, \ldots, \omega_{n-k}, \phi_j \in \mathrm{PSH}(U, \omega_j) \cap L^\infty(X)$, and set*

$$T := (\omega_1 + dd^c \phi_1) \wedge \ldots \wedge (\omega_{n-k} + dd^c \phi_{n-k}).$$

Then we have

$$1_{\{u>v\}} (\omega + dd^c \max(u,v))^k \wedge T = 1_{\{u>v\}} (\omega + dd^c u)^k \wedge T.$$

[8] This doesn't change at all the argument because we could have chosen it like this when we described what f we wanted for $\mathcal{T}(v) = f$. However, it would make the calculations slightly worst although analogous.

If moreover $u \leqslant v$ in U, then

$$1_{\{u=v\}}\left(\omega+dd^c u\right)^k \wedge T \leqslant 1_{\{u=v\}}\left(\omega+dd^c v\right)^k \wedge T.$$

Proof. The first equality comes from a usual argument in pluripotential theory. (see Theorem 3.27 in [GZ17].)

To prove the second inequality we consider $\psi_t := \max(u+t, v), t>0$. The first equation can be seen as an inequality if you consider one side on a bigger domain. Then we get:

$$\left(\omega+dd^c \psi_t\right)^k \wedge T \geqslant 1_{\{u+t>v\}}\left(\omega+dd^c u\right)^k \wedge T$$

Letting $t \to 0^+$ we obtain $\left(\omega+dd^c \max(u,v)\right)^k \wedge T \geqslant 1_{\{u=v\}}\left(\omega+dd^c u\right)^k \wedge T$, which gives the result.

□

Now we prove a central result in Pluripotential Theory, the Comparison Principle. In the Hermitian setting, however we can't have a Comparison Principle as known in the Kähler case, the best we can get is a *modified comparison principle.* More precisely according to the Proposition 9.1 in [Din19] states that a necessary condition for the usual Comparison Theorem to work it is needed that the volume of the dd^c pertubation is constant. For more on when this volume is constant see the comments after Definition 2.2. For the original reference see Kolodziej-Nguyen [KN15][Theorem 0.2]:

Theorem 4.3. *Let u, v be bounded ω-psh functions. For $\lambda \in (0,1)$ we set $m_\lambda = \inf_X\{u-(1-\lambda)v\}$. Then*

$$\left(1-\frac{4B(n-1)^2 s}{\lambda^3}\right)^n \int_{\{u<(1-\lambda)v+m\lambda+s\}} w^n_{(1-\lambda)v} \leqslant \int_{\{u<(1-\lambda)v+m\lambda+s\}} \omega_u^n.$$

for all $0<s<\frac{\lambda^3}{32B(n-1)^2}$.

the proof by Kołodziej-Nguyen relies on the main result of [DK12] and some extra technical estimates. Here we present a simplified proof coming from [Lu21].

Proof. Take $\phi := \max\left(u,(1-\lambda)v+m_\lambda+s\right), U_{\lambda,s} := \{u<(1-\lambda)v+m_\lambda+s\}$. For $0 \leqslant k \leqslant n$ we set $T_k := \omega_u^k \wedge \omega_\phi^{n-k}$, and $T_l = 0$ if $l<0$. Set $a = Bs\lambda^{-3}(n-1)^2$. We will proceed the proof inductively. We will prove for $k = 0, 1, \ldots, n-1$ that:

$$(1-4a)\int_{U_{\lambda,s}} T_k \leqslant \int_{U_{\lambda,s}} T_{k+1} \tag{4.1}$$

The conclusion follows since $(\omega_\phi)^n = \left(\omega_{(1-\lambda)v}\right)^n$ in the set $U_{\lambda,s}$.[9] This is a direct applicatin of Lemma 4.2.

We first prove 4.1 for $k=0$:

- $k=0$: Since $u \leqslant \phi$, Lemma 4.2 ensures that

  $$1_{\{u=\phi\}}\omega_\phi^n \geqslant 1_{\{u=\phi\}}\omega_u \wedge \omega_\phi^{n-1}$$

  Note that $U_{\lambda,s} = \{u<\phi\}$, we now have

  $$\int_X dd^c(\phi-u)\wedge\omega_\phi^{n-1} = \int_X\left(\omega_\phi^n-\omega_u\wedge\omega_\phi^{n-1}\right) \geqslant \int_{U_{\lambda,s}}\omega_\phi^n-\int_{U_{\lambda,s}}\omega_u\wedge\omega_\phi^{n-1}.$$

  Now we will use the above inequality to get the estimate we need using the constant from the condition (B) (see Definition 2.2). We plan to use Stokes theorem to arrive at the calculations related to the condition (B). Then we do the following calculations:

[9] A very important comment here is that this set is not necessary open, because the functions in question are not necessary continous. These sets can be seen as opens on the Plurifine Topology, the interested reader can reference to [Wi12] for details around this topic.

$$
\begin{aligned}
dd^c\omega_\phi^{n-1} &= (n-1)dd^c\omega\wedge\omega_\phi^{n-2} + (n-1)(n-2)d\omega\wedge d^c\omega\wedge\omega_\phi^{n-3}\\
&\leqslant (n-1)B\omega^2\wedge\omega_\phi^{n-2} + (n-1)(n-2)B\omega^3\wedge\omega_\phi^{n-3}
\end{aligned}
$$

since ω satisfies condition (B). As $\phi - u \geqslant 0$, by definition, it follows from sucessive aplications of Stokes theorem (see Lemma B.5) that

$$
\begin{aligned}
&\int_X dd^c(\phi-u)\wedge\omega_\phi^{n-1}\\
&\leqslant (n-1)B\left\{\int_X(\phi-u)\omega^2\wedge\omega_\phi^{n-2} + (n-2)\int_X(\phi-u)\omega^3\wedge\omega_\phi^{n-3}\right\}.
\end{aligned}
$$

Now we want to estimate de integrals on the right hand side from above, for that note:
– As $0 \leqslant (1-\lambda)\omega_v$ we get $\lambda\omega \leqslant \omega_{(1-\lambda)v}$ hence $\omega^j\wedge\omega_\phi^{n-j} \leqslant \lambda^{-j}\left(\omega_{(1-\lambda)v}\right)^j\wedge\omega_\phi^{n-j}$.
– By Lemma 4.2, $\left(\omega_{(1-\lambda)v}\right)^j\wedge\omega_\phi^{n-j} = \omega_\phi^n$ in the set $U_{\lambda,s}$.
– By construction $0 \leqslant \phi - u \leqslant s$ and $\phi - u = 0$ on $X\backslash U_{\lambda,s}$.
With these three facts we conclude: $\int_X(\phi-u)\omega^j\wedge\omega_\phi^{n-j} \leqslant s\lambda^{-j}\int_{U_{\lambda,s}} w_\phi^n$, for $j = 2,3$, hence

$$
\int_{U_{\lambda,s}}\omega_\phi^n - \int_{U_{\lambda,s}}\omega_u\wedge\omega_\phi^{n-1} \leqslant \int_X dd^c(\phi-u)\wedge\omega_\phi^{n-1} \leqslant \frac{Bs(n-1)^2}{\lambda^3}\int_{U_{\lambda,s}}\omega_\phi^n,
$$

since $\lambda^{-2} \leqslant \lambda^{-3}$. This yields the Inequality 4.1 for $k = 0$.

- *inductive argument:* We asume now that the Inequality 4.1 holds for all $j \leqslant k-1$, and we check that it still holds for k. Note that

$$
\begin{aligned}
&dd^c\left(\omega_u^k\wedge\omega_\phi^{n-[k+1]}\right)\\
=&kdd^c\omega\wedge\omega_u^{k-1}\wedge\omega_\phi^{n-[k+1]} + (n-[k+1])dd^c\omega\wedge\omega_u^k\wedge\omega_\phi^{n-[k+2]}\\
&+2k(n-[k+1])d\omega\wedge d^c\omega\wedge\omega_u^{k-1}\wedge\omega_\phi^{n-[k+2]}\\
&+k(k-1)d\omega\wedge d^c\omega\wedge\omega_u^{k-2}\wedge\omega_\phi^{n-[k+1]}\\
&+(n-[k+1])[n-(k+2)]d\omega\wedge d^c\omega\wedge\omega_u^k\wedge\omega_\phi^{n-[k+3]}.
\end{aligned}
$$

Analogously as before we get

$$
\begin{aligned}
\int_{U_{\lambda,s}}(T_k - T_{k+1}) \leqslant \int_X(T_k - T_{k+1}) &= \int_X(\phi-u)dd^c\left(\omega_u^k\wedge\omega_\phi^{n-[k+1]}\right)\\
&\leqslant \frac{Bs}{\lambda^3}\int_{U_{\lambda,s}}\left(k(k-1)T_{k-2} + 2k[n-k]T_{k-1} + (n-[k+1])^2T_k\right)\\
&\leqslant a\left(\frac{1}{(1-4a)^2} + \frac{1}{1-4a} + 1\right)\int_{U_{\lambda,s}}T_k \leqslant 4a\int_{U_{\lambda,s}}T_k
\end{aligned}
$$

the third inequality above we have used the induction hypothesis. The fourth inequality follows from the upper bound $4a < 1/8$ and the choice of s in the hypothesis of the Theorem. From this we obtain Inequality 4.1 for k, finishing the proof.

□

We now establish the following generalization of the domination principle:[10]

Proposition 4.4. *Fix $c \in [0,1)$. If u, v are bounded w-psh functions such that $\omega_u^n \leqslant cw_v^n$ on $\{u < v\}$, then $u \geqslant v$.*

[10]The classical domination principle is when $c = 0$ (see [Ngu16]).

Proof. Up to adding the same constant to both u and v we can assume without lost of generality $v \geqslant 0$. Assume by contradiction that $\inf_X(u-v) < 0$. Now, we will take constants λ, s to apply the Comparison Principle constructed to arrive at a contradiction. Fix $\lambda > 0$ (using $c \in [0,1)$ and the assumption) so small that

$$m_\lambda := \inf_X (u - (1-\lambda)v) < 0 \text{ and } (1-\lambda)^n > c$$

Fixing a small constant $s > 0$ (that will be made precise how small later), by the comparison principle (Theorem 4.3) we have

$$\left(1 - \frac{4B(n-1)^2 s}{\lambda^3}\right)^n \int_{\{u<(1-\lambda)v+m_\lambda+s\}} \omega^n_{(1-\lambda)v} \leqslant \int_{\{u<(1-\lambda)v+m_\lambda+s\}} \omega^n_u.$$

We take s is small enough to have both $\{u < (1-\lambda)v + m_\lambda + s\} \subset \{u < v\}$ and

$$\left(1 - \frac{4B(n-1)^2 s}{\lambda^3}\right)^n (1-\lambda)^n > c$$

Note that, by the binomial expansion of $\omega_{(1-\lambda v)}$ and the fact that for λ small enough we have $(1-(1-\lambda)^n) \geqslant \lambda^n$, we have $\omega^n_{(1-\lambda)v} \geqslant \lambda^n \omega^n + (1-\lambda)^n \omega^n_v$. Combining the above inequalities and the assumption that $w^n_u \leqslant c w^n_v$ on $\{u < v\}$, we obtain

$$\int_{\{u<(1-\lambda)v+m_\lambda+s\}} \omega^n = 0$$

which is a contradiction. □

Corollary 4.5. *Let u, v be bounded ω-psh functions.*

(i) *If $\omega^n_u \leqslant c\omega^n_v$ for some $c \geqslant 0$ then $c \geqslant 1$.*
(ii) *If $\omega^n_u \geqslant e^{a(u-v)}\omega^n_v$ for some $a > 0$ then $u \leqslant v$.*

Proof. The first affirmation comes form the assuming by contradiction $c \in [0,1)$ and substitute v for $v + \underset{X}{sup}|u| + \underset{X}{sup}|v|$. This gives the contradiction of u being bigger then it's *sup*.

The second comes from the calculation:

$$\omega^n_u \geqslant e^{a(u-v)}\omega^n_v \geqslant e^{a(\inf(u-v))}\omega^n_v \Leftrightarrow \frac{1}{c}\omega^n_u \geqslant \omega^n_v \underset{(i)}{\Rightarrow} \frac{1}{c} \geqslant 1 \Rightarrow u \leqslant v.$$

□

Now we are ready to prove the L^∞ estimate. We will first prove a technical Lemma and then get the estimate as a corollary of it. This approach comes from [Lu21, GL21].

It is relevant to notice that for our case we don't need all the generality of these results because our density f is smooth, positive on a compact manifold. However, assuming the uniform estimate from [Kol98] that we use in the proof, the extra generality of the Lemma 4.6 and Theorem 4.7 offers no extra dificulty on the proof.

Lemma 4.6. *Fix $p > 1$. There exists constants c_p, C_p depending on p, X, ω such that the following holds: if $0 \leqslant f \in L^P(X)$ with $||f||_p \leqslant 1$, then there exists $u \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ such that $(\omega + dd^c u)^n \geqslant c_p f dV$ and $\mathrm{osc}_X(u) \leqslant C_p$.*

Proof. By compactness we take finite double cover of X such that we have the finite cover of balls $(B_j)_{j=1}^N$ such that $B_j \Subset B'_j \Subset U_j$ and (U_j, ψ_j) is a holomorphic chart of X. We can take the covering such that, under this biholomorphism, B'_j is mapped to the unit ball in $\mathbb{C}^n$, while B_j is mapped to the ball of radius $1/2$. Now the idea will be that locally we can solve these by using the local theory, however the gluing process shouldn't work directly because of the Hermitian setting. With that in mind we will use the solution on the bigger ball B'_j, restrict it to the smaller ball B_j and patch things together outside of it with a function that behaves like $log|z|$ so that we understnd well the behavior globaly and we removed the singularity. Then smooth things out, outside the sets of our construction, using a cutoff function. This will allow us to define we a global psh as all consructions will preserve the property of being psh up to renormalization. Then, we'll get a $PSH_\omega(X)$ function with the wanted properties.

First for each B'_j we solve the complex Monge-Ampère equation (see [Kol98] for the uniform estimate):[11]

$$(dd^c v_j)^n = f dV \text{ in } B'_j, v_j = -1 \text{ on } \partial B'_j.$$

Then $|v_j| \leqslant A \log 2$ for a uniform constant $A > 0$ depending on p, B'_j. To start the gluing process we take $u_j := \max(v_j, A\log|z|)$. Then in $\{|z| > e^{-1/A}\}$ we have because of the boundary condition $u_j = A\log|z|$, while because of the bound $u_j = v_j$ in B_j. We next extend u_j to a smooth function outside B'_j by multiplying it with a cutoff function. This gives us a function which behaves well globaly on X and which is ω-psh in B'_j. Also, uniformily bounded because it is smooth where it might not be ω-psh. This also implies that up to a multiplicative uniform constant $a > 0$, we can control the curvature $dd^c u_j$ making it small enough.

With that, we obtain a function still denoted by u_j with the following properties: $\omega + dd^c(au_j) \geqslant 0$ on X and $(\omega + dd^c au_j)^n \geqslant a^n f dV$ in B_j. Where the last inequality comes simply from choosing small a to have $(\omega + dd^c au_j)^n \geqslant a^n(\omega + dd^c u_j)^n$

We then define $u := N^{-1}\sum_{j=1}^N au_j \in \mathrm{PSH}(X,\omega)$ and because our indexes are all finite we have

$$(\omega + dd^c u)^n \geqslant cf dV \text{ on } X,$$

where $c > 0$ depends on p, N, B_j, B'_j. Giving us the bounded subsolution we wanted.

□

Now we get the actual L^∞ estimate:

Theorem 4.7. *Assume $0 \leqslant f \in L^p(X)$ with $||f||_p \leqslant C$ and $C^{-1} \leqslant ||f||_{1/n}$. Let $(u,c) \in \mathrm{PSH}(X,\omega) \cap L^\infty(X) \times \mathbb{R}^+$be a solution to the Monge-Ampere equation*

$$(\omega + dd^c u)^n = cf dV$$

Then c, c^{-1}, and $\mathrm{oscX}(u)$ are bounded from above by a constant depending on C, p, X, ω, n.

Proof. We normalize u by $\sup_X u = 0$. Fix $\epsilon > 0$ so small that $g := e^{-\epsilon u} f \in L^q(X)$ for some $q > 1$. We can do that, independent of u, by Skoda's Uniform Integrability (see [GZ17] Theorem 8.11). It follows from Lemma 4.6 that there exists a bounded w-psh function v such that

$$(\omega + dd^c v)^n \geqslant c_q \frac{g}{||g||_q} dV \geqslant \frac{c_q}{||g||_q} f dV,$$

with a uniform bound $-C_q \leqslant v \leqslant 0$. The domination principle, Corollary 4.5 , yields $c \geqslant c_q ||g||_q^{-1}$. The upper bound for c follows from the mixed Monge-Ampère inequality (see [Din09] or Lemma 1.9 in [Ngu16]).

$$C_1 \geq \int_X \omega_u \wedge \omega^{n-1} \geqslant \int_X c^{1/n} f^{1/n} \omega^n.$$

From

$$(\omega + dd^c v)^n \geqslant e^{a(v - C_2 - u)} cf dV$$

and the domination principle (Corollary 4.5), we obtain $v - C_2 \leqslant u$, hence $\sup_X |u|$ is also uniformly bounded.

□

[11]In this proof we will be using interchangibly the neighborhoods/functions on the manifold and the coresponding ones on $\mathbb{C}^n$. This is done not to overload the notation.

5. Laplacian Estimate

Here is where we prove the estimate on the Laplacian (Theorem 5.2). We use a type of canonical coordinates introduced by Guan-Li [GuaLi10] and with it we can simplify a lot of calculations to relate the geometry and the laplacian of a solution to the Monge-Ampère. This is certainly the most demanding part in terms of calculations, but not in technical knowledge. Our approach here comes from [Din19, GuaLi10].

Before proving the Δ-estimate we will discuss some preliminary facts first.

First, I will make clear that whenever the word Laplacian appear it is meant as the Chern Laplacian[12] related to the metric in question. Namely, for a function f,

$$\Delta f = g^{i\bar{j}}\partial_i\partial_{\bar{j}} f = \frac{n\omega^{n-1}\wedge dd^c f}{\omega^n}$$

Note that $\operatorname{tr}_g g' = n + \Delta u$, where $g' = g_{i\bar{j}} + u_{i\bar{j}}$.

We start with an example of a function which has bounded Laplacian but is not C^2, hence the Laplacian is bounded but not the full Hessian. This comes to ilustrate that although this estimates that we seek gives us a bound on second derivatives it does not imply full control over those. [13] The function w bellow has continuous laplacian but unbounded Hessian:

$$w(x,y) = \begin{cases} \left(x^2-y^2\right)\ln\left(-\ln\left(x^2+y^2\right)\right) & 0 < x^2+y^2 \leqslant \frac{1}{4} \\ 0 & (x,y) = (0,0) \end{cases}$$

Although we have examples as above, we will get on Section 6 that a Laplacian estimate is enough to proceed with our argument and reach higher regularities and their estimates.

In the Kähler setting we have the normal coordinates. Which diagonalizes the metric at the point (zeroth-order) and makes the first derivative terms of the metric zero (first order) and the second order terms dependent on the curvature. (as in classical Riemannian Geometry.) In the Hermitian setting we can't have such a good simplification to our calculations, however we can find some simplifications that might suit the problem we try to solve. For these types of arguments we pick a reference point p and look at coordinates around them. We identify this point with the zero of the chart and try to analyze and simplify the expansion of the metric around it. This gives us that for local calculations that don't use any derivative we can just take our metric as being its zeroth order term, for the ones that use only one derivative we expand until the first order, and so on and so forth. This helps us not only find the simplification of our metric around p but also apply the simplification to make calculations easier. Calculations such as the one on the estimate of Theorem 5.2.

Now we present the coordinates of Guan-Li that will be used in the proof.

Proposition 5.1. [GuaLi10]

Given a Hermitian manifold (X,ω) and a point $p \in X$ it is possible to choose coordinates near p, such that $g_{i\bar{j}}(p) = \delta_{ij}$ and for any pair i,k one has $\frac{\partial g_{i\bar{i}}}{\partial z_k}(p) = 0$.

Proof. As commented above, we take local coordinates z_i around p. We can take it such that at this point the metric is diagonalized. Then choose new coordinates from this first one by adding some quadratic terms:

$$w_r = z_r + \sum_{m\neq r} \frac{\partial g_{r\bar{r}}}{\partial z_m} z_m z_r + \frac{1}{2}\frac{\partial g_{r\bar{r}}}{\partial z_r} z_r^2.$$

Note that, at P we have:

[12] Laplacian related to the Chern Connection.

[13] For a general construction regarding examples of this fashion and analogous on higher order derivatives see [PY23].

$$\frac{\partial z_r}{\partial w_i} = \delta_{ri}$$
$$\frac{\partial^2 z_r}{\partial w_i \partial w_k} = -\sum_{m\neq r} \frac{\partial g_{r\bar{r}}}{\partial z_m}\left(\frac{\partial z_m}{\partial w_i}\frac{\partial z_r}{\partial w_k} + \frac{\partial z_m}{\partial w_k}\frac{\partial z_r}{\partial w_i}\right) - \frac{\partial g_{r\bar{r}}}{\partial z_r}\frac{\partial z_r}{\partial w_i}\frac{\partial z_r}{\partial w_k}.$$

Now we analyse the metric with respect to our new coordinates: $\tilde{g}_{i\bar{j}} := g\left(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_j}\right)$, this leads us to:

$$\begin{aligned}\frac{\partial \tilde{g}_{i\bar{j}}}{\partial w_k} &= \sum_{r,s=1}^{n} g_{r\bar{s}} \frac{\partial^2 z_r}{\partial w_i \partial w_k}\frac{\partial \bar{z}_s}{\partial \bar{w}_j} \\ &\quad + \sum_{r,s,p=1}^{n} \frac{\partial g_{r\bar{s}}}{\partial z_p}\frac{\partial z_p}{\partial w_k}\frac{\partial z_r}{\partial w_i}\frac{\partial \bar{z}_s}{\partial \bar{w}_j}.\end{aligned}$$

Using the equations that relate z_i with w_k and remembering $g_{i,\bar{j}}(p) = \delta_{ij}$ we get:

$$\begin{aligned}\frac{\partial \tilde{g}_{i\bar{i}}}{\partial w_k} =& \sum_{r=1}^{n}\left(-\sum_{m\neq r} -\frac{\partial g_{r\bar{r}}}{\partial z_m}\left(\delta_{mi}\delta_{rk} + \delta_{mk}\delta_{ri}\right)\delta_{ri} - \frac{\partial g_{r\bar{r}}}{\partial z_r}\delta_{ri}\delta_{rk}\right) \\ &+ \sum_{r,s,p=1}^{n} \frac{\partial g_{r\bar{s}}}{\partial z_p}\delta_{pk}\delta_{ri}\delta_{si} = 0.\end{aligned}$$

□

Now we have the ingredients to prove the estimate. The Laplacian estimate we want to prove is:

Theorem 5.2. [GuaLi10]
If u is a solution to the MA problem on Theorem 2.8 then there exists a constant $C = C\left(X, \omega, n, ||\Delta f||_{C^0}, ||u||_{C^0}\right)$, such that

$$0 \leqslant n + \Delta u \leqslant C$$

where the Laplacian is the ordinary Chern Laplacian with respect to the metric ω.
The idea of the proof comes from [GuaLi10] however the structure in it self comes from [Din19].

Proof. The classical idea that comes from the Kähler case (see [Yau78, Aub78] or in more generality [GZ17] Section 14.2) is to study a function such as $A(u) := \log(n + \Delta u) + h \circ u$, where h is an additional uniformly bounded strictly decreasing function that we shall choose later on. The idea is that analysing the maximum of this function through derivatives should give us bounds for the trace $n + \Delta u$ and consequentely for the Laplacian itself. Proving that at the point z where A attains maximum we have a bound for $n + \Delta u$ then proof is finished. Since at any other point x we have

$$\log(n + \Delta u)(x) \leqslant A(z) - h(u(x)) \leqslant C$$

Fixing a point z of maximum of A and identify it with zero in a local chart. We will use the idea commented earlier in this section of calculations with derivatives on a chart and the expansion of the metric. In particular $g_{i\bar{j},k}$ will denote $\frac{\partial g_{i\bar{j}}}{\partial z_k}$. We will also denote g' the metric $g_{i\bar{j}} + u_{i\bar{j}}$, while $g^{k\bar{l}}, g'^{k\bar{l}}$ will denote the inverse transposed matrices of g and g' respectively.
To reduce our calculations we will use the coordinate system of Guan-Li on Proposition 5.1. Note the Hessian of u is still diagonal at zero. We can assume that $\Delta u(0) \geqslant 1$, because otherwise the proof is over.

Applying logarithm to both sides of MA Equation on Theorem 2.8[14] and differentiating twice at z we get:[15]

$$g'^{p\bar{r}}\left(g_{p\bar{r},k}+u_{p\bar{r}k}\right)=\log(f)_k+g^{p\bar{r}}g_{p\bar{r},k} \tag{5.1}$$

$$\begin{aligned}-g'^{p\bar{s}}g'^{h\bar{r}}\left(g_{h\bar{s},\bar{l}}+u_{h\bar{s}\bar{l}}\right)\left(g_{p\bar{r},\bar{k}}+u_{p\bar{r}\bar{k}}\right)+g'^{p\bar{r}}\left(g_{p\bar{r},k\bar{l}}+u_{p\bar{r}k\bar{l}}\right)\\ =\log(f)_{k\bar{l}}-g^{p\bar{s}}g^{h\bar{r}}g_{h\bar{s},\bar{l}}g_{p\bar{r},k}+g^{p\bar{r}}g_{p\bar{r},k\bar{l}}\end{aligned} \tag{5.2}$$

To relate the above identities with the trace we want to estimate we take trace in the second equation above:

$$-g'^{p\bar{p}}g'^{r\bar{r}}\left|g_{r\bar{p},k}+u_{r\bar{p}k}\right|^2+g'\bar{r}\bar{r}\left(g_{r\bar{r},k\bar{k}}+u_{r\bar{r}k\bar{k}}\right)=\Delta\log(f)-\left|g_{p\bar{r},k}\right|^2+g_{r\bar{r},k\bar{k}} \tag{5.3}$$

With these identities that relate the solution and the metric we now look A at z where it achieves its maximum. From the vanishing of the first derivative of A we get the equalities:

$$0=\frac{g^{i\bar{j}}_{,k}u_{i\bar{j}}+g^{i\bar{j}}u_{i\bar{j}k}}{\Delta u+n}+h'u_k=\frac{u_{i\bar{i}k}}{\Delta u+n}+h'u_k \tag{5.4}$$

Note that the first term vanishes from the first to the second equation because of the Guan-Li coordinates and the Hessian of u being diagonal at zero. Now we take the trace of the Hessian of A at z relative to g' and get:

$$0\geqslant g'^{k\bar{k}}A_{k\bar{k}}=g'^{k\bar{k}}\left[\frac{\left(g^{i\bar{j}}u_{i\bar{j}}\right)_{k\bar{k}}}{\Delta u+n}-\frac{\left|\sum_i u_{i\bar{i}k}\right|^2}{(\Delta u+n)^2}+h'u_{k\bar{k}}+h''\left|u_k\right|^2\right]. \tag{5.5}$$

From Eq. 5.4 the second term of the above expression can be simplified to $-\left(h'\right)^2g'^{k\bar{k}}\left|u_k\right|^2$, considering the distributive fo the metric term on the sum. The third one can be simplified as $h'\left(n-\sum_k g'k\bar{k}\right)$, by taking the trace with respect to $g'^{i\bar{j}}$ of the definition of g'.

In order to estimate the first term we will have to open it and do a couple of substitutions to get the dependence on the factors we want. First we have the direct computation:

$$\left(g^{i\bar{j}}u_{i\bar{j}}\right)_{k\bar{k}}=g^{i\bar{i}}_{,k\bar{k}}u_{i\bar{i}}+u_{i\bar{i}k\bar{k}}+2\operatorname{Re}\left(g^{i\bar{j}}_{k}u_{i\bar{j}\bar{k}}\right)$$

The term with four derivatives, after taking trace with $g'^{k\bar{k}}$ can be exchanged using Eq. 5.3. Note that, as g is diagonal at z we get

$$g^{i\bar{j}}_{,k}=-g^{i\bar{s}}g^{l\bar{j}}g_{l\bar{s},k}=-g_{j\bar{i},k}$$

Gathering both relations above we rewrite the first term of Ineq. 5.5 as:

$$\begin{aligned}g'^{k\bar{k}}\frac{\left(g^{i\bar{j}}u_{i\bar{j}}\right)_{,k\bar{k}}}{\Delta u+n}=g'^{k\bar{k}}\frac{g^{i\bar{i}}_{k\bar{k}}u_{i\bar{i}}}{\Delta u+n}-g'^{k\bar{k}}\frac{2\operatorname{Re}\left(g_{j\bar{i},k}u_{i\bar{j}\bar{k}}\right)}{\Delta u+n}-g'^{k\bar{k}}\frac{g_{k\bar{k},i\bar{i}}-\Delta\log f}{\Delta u+n}\\ -\frac{\left|g_{r\bar{k},i}\right|^2}{\Delta u+n}+\frac{g'^{r\bar{r}}g'^{k\bar{k}}\left|g_{r\bar{k},i}+u_{r\bar{k}i}\right|^2}{\Delta u+n}\end{aligned}$$

Note that the first summand above is controlled from below by $-C\sum_k g'^{k\bar{k}}$ with the constant C dependent on the sup norm of all second order derivatives of g. Which comes from a calculation quite similar to the one to simplify Ineq 5.5. The same goes for all the terms in the third and fourth summand.[16]

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[14]Note we are not estimating solutions to the continuity method but to the original MA Equation, however doing it for the conitnuity equation produces no extra dificulties besides complicated notation.

[15]This calculation is a global version of the one done in the beginning of the proof of Theorem 6.1.

[16]The fact that the is not $u_{i\bar{i}}$ term on the numerator means that the estimate could be even sharper, but we don't need it.

Summing up our computations up to now with Inequality 5.5 we get

$$0 \geqslant \left[-h' - C\right] \sum_k g'^{k\bar{k}} - \widetilde{C} + \left[h'' - \left(h'\right)^2\right] \sum_k g'^{k\bar{k}} \left|u_k\right|^2 + \frac{g'^{r\bar{r}} g'^{k\bar{k}} \left|g_{r\bar{k},i} + u_{r\bar{k}i}\right|^2}{\Delta u + n}$$
$$- g'^{k\bar{k}} \frac{2\operatorname{Re}\left(g_{j\bar{i},k} u_{i\bar{j}\bar{k}}\right)}{\Delta u + n}.$$

A very relevant fact is that $\widetilde{C}$ represents a constant which depends on h, in this case more specifically on a uniform bound on h'. A understanding of this bound and constant will be clear at the end once we choose our function.

The last term above can be rewritten as follows:

$$g'^{k\bar{k}} \frac{2\operatorname{Re}\left(g_{j\bar{i},k} u_{i\bar{j}\bar{k}}\right)}{\Delta u + n} = g'^{k\bar{k}} \frac{2\operatorname{Re}\left(g_{j\bar{i},k} u_{i\bar{k}\bar{j}}\right)}{\Delta u + n} = g'^{k\bar{k}} \frac{2\operatorname{Re}\left(g_{j\bar{i},k}\left(g_{i\bar{k},\bar{j}} + u_{i\bar{k}\bar{j}} - g_{i\bar{k},\bar{j}}\right)\right)}{\Delta u + n}$$
$$= g'^{k\bar{k}} \sum_{i\neq j} \sqrt{g'^{i\bar{i}} g'_{i\bar{i}}} \frac{2\operatorname{Re}\left(g_{j\bar{i},k} g'_{i\bar{k},\bar{j}}\right)}{\Delta u + n} - g'^{k\bar{k}} \frac{2\operatorname{Re}\left(g_{j\bar{i},k} g_{i\bar{k},\bar{j}}\right)}{\Delta u + n}.$$

Notice we sum only over indices $i \neq j$ because the Guan-Li coordinates satisfy $g_{i\bar{i},k} = 0$. Applying Schwarz inequality[17] the latter is bounded above by

$$g'^{k\bar{k}} \sum_{i\neq j} g'^{i\bar{i}} \frac{\left|g_{i\bar{k},\bar{j}}\right|^2}{\Delta u + n} + g'^{k\bar{k}} \sum_{i\neq j} \frac{g'_{i\bar{i}} \left|g_{j\bar{i},k}\right|^2}{n + \Delta u} + C \sum_k g'^{k\bar{k}} \leqslant \sum_{i\neq j} g'^{k\bar{k}} g'^{i\bar{i}} \frac{\left|g'_{i\bar{k},\bar{j}}\right|^2}{\Delta u + n}$$
$$+ C \sum_k g'^{k\bar{k}}$$

With this we can reduce our main inequality to

$$0 \geqslant \left[-h' - C\right] \sum_k g'^{k\bar{k}} - \widetilde{C} + \left[h'' - \left(h'\right)^2\right] \sum_k g'^{k\bar{k}} \left|u_k\right|^2 + \frac{g'^{r\bar{r}} g'^{k\bar{k}} \left|g'_{r\bar{k},k}\right|^2}{\Delta u + n}$$

The last term can be handled as follows

$$\frac{g'^{r\bar{r}} g'^{k\bar{k}} \left|g'_{r\bar{k},k}\right|^2}{\Delta u + n} = g'^{r\bar{r}} \frac{\left[\left(\sum_k g' k\bar{k} \left|g'_{r\bar{k},k}\right|^2\right)\left(\sum_k g'_{k\bar{k}}\right)\right]}{(\Delta u + n)^2} \geqslant g'^{r\bar{r}} \frac{\left|\sum_k \left(u_{r\bar{k}k} + g_{r\bar{k},k}\right)\right|^2}{(\Delta u + n)^2}$$
$$= g'^{r\bar{r}} \left|h' u_r + \frac{\sum_k g_{r\bar{k},k}}{\Delta u + n}\right|^2,$$

In the last equality we use Eq. 5.4. Expanding the squares and applying Schwarz inequality again we get:

$$\frac{g'^{r\bar{r}} g'^{k\bar{k}} \left|g'_{r\bar{k},k}\right|^2}{\Delta u + n} \geqslant g'^{r\bar{r}} \left(\left(h'\right)^2 + h'\right) \left|u_r\right|^2 - \left|h'\right| g'^{r\bar{r}} \frac{\left|\sum_k g_{r\bar{k},k}\right|^2}{(\Delta u + n)^2},$$

again we can estimate last summand by $C\widetilde{C} \sum_r g'^{r\bar{r}}$. Where we write $\widetilde{C}$ because there is a factor inside it that comes from estimating $|h'|$ uniformilly, which will be clear why we can by the choice of h later.[18]

Putting the pieces together we get our main inequality as:

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[17]and also the elementary inequality $g'_{i\bar{i}} \leqslant \Delta u + n$.

[18]Here we use the usual notation that a constant such as C or $\widetilde{C}$ can vary from line to line, but on what the constant depends is the most important and is fixed from line to line.

$$0 \geqslant \left[-h' - C_1\widetilde{C}_1\right] \sum_k g'^{k\bar{k}} - \widetilde{C}_2 + \left[h'' + h'\right] \sum_k g'^{k\bar{k}} \left|u_k\right|^2$$

So if we choose the function $h(t) = Ke^{-t}$. Then we can consider h as having uniform bound as well as h' and both bounds related, because $\underset{X}{osc}(u)$ is uniformilly bounded by Theorem 4.7.

Now this gives us something that behaves like

$$0 \geqslant (\widetilde{C}_3 - \widetilde{C}_1 C_1) \sum_k g'^{k\bar{k}} - \widetilde{C}_2$$

which shows that $g'k\bar{k}$ are upper bounded and hence $g'_{k\bar{k}}$ are also lower bounded. From the equation we immediately get that $g'_{k\bar{k}}$ are upper bounded at the point z which establishes the desired estimate.

□

6. Higher Order Estimates

This the part of the argument is local in nature due to the *Complex Evans-Krylov Theory*. See [GT83] for references on the real case and [Siu87] for the original approach in the complex case. The argument here will consist in using this Complex Evans-Krylov Theory to use the L^∞ *− estimate and* Δ *− estimate* to get a $C^{2,\alpha}$ *− estimate* (Theorem 6.1). Then use the usual Schauder Theory for Second Order Elliptic PDEs with a bootstraping argument to get as much regularity as we can as in Theorem 6.2, in our case we'll get smoothness. Assuming familiarity with PDEs, specially Schauder Theory this should be the most straightforward part. Our approach comes from [Blo12, GZ17].

In this chapter we want to prove the following result:

Theorem 6.1. *Let u be a $\mathcal{C}^4$-smooth plurisubharmonic function in an open bounded connected set $\Omega \subset \mathbb{C}^n$ such that* $\det\left(u_{i\bar{j}}\right) = e^g > 0$*. Then for any $\Omega' \Subset \Omega$ there exists*

$$0 < \alpha = \alpha\left(n, ||u||_{\mathcal{C}^{0,1}(\Omega)}, \sup_\Omega \Delta u, ||g||_{\mathcal{C}^{0,1}(\Omega)}, \inf_\Omega g\right) < 1,$$

and $C > 0$, depending moreover on a lower bound for $\operatorname{dist}(\Omega', \partial\Omega)$*, such that*

$$||u||_{\mathcal{C}^{2,\alpha}(\Omega')} \leqslant C$$

Proof. We start our proof presenting some classical calculations.[19] See Exercise 14.13 in [GZ17] for a step by step for those.

Fix $\xi \in \mathbb{C}^n, |\xi| = 1$. We will differentiate our local problem with respect to this direction:

$$\log\det\left(u_{i\bar{j}}\right) = \log f = g$$

We get

$$u^{i\bar{j}} u_{\xi\bar{\xi}i\bar{j}} = (\log f)_{\xi\bar{\xi}} + u^{i\bar{l}} u^{k\bar{j}} u_{\xi i\bar{j}} u_{\bar{\xi}k\bar{l}}$$

We reinforce that we use u^{ij} for the inverse transpose of u_{ij}, as in Theorem 5.2. Now we use the concavity of $A \to \log\det A$ (see Lemma B.3) where A is a positive $n \times n$ Hermitian matrix.

This gives us the inequality $u^{i\bar{l}} u^k \bar{j} u_{\xi i\bar{j}} u_{\bar{\xi}k\bar{l}} \geqslant 0$ and with that we get:

$$u^{i\bar{j}} u_{\xi\bar{\xi}i\bar{j}} \geqslant (\log f)_{\xi\bar{\xi}} \tag{6.1}$$

To use techniques from [GT83] we will write the Inequality 6.1 in divergence form. Le $a^{i\bar{j}} := fu^{i\bar{j}}$. For any fixed i we have:

[19] This calculation is a local version of the one done in the beginning of the proof of Theorem 5.2.

$$\left(a^{i\bar{j}}\right)_{\bar{j}} = f\left(u^{i\bar{j}}u^{k\bar{l}} - u^{i\bar{l}}u^{k\bar{j}}\right)u_{k\bar{l}\bar{j}} = 0$$

now applying Ineq. 6.1 and the since $f \in C^{0,1}$,

$$\left(a^{i\bar{j}}u_{\xi\bar{\xi}i}\right)_{\bar{j}} \geqslant f_{\xi\bar{\xi}} - \frac{|f_\xi|^2}{f} \geqslant f_{\xi\bar{\xi}} - C_1.$$

Rewriting in gradient form we will get

$$f_{\xi\bar{\xi}} = \sum_{i,j} c_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j} = \sum_{i,j} c_{i,j}\frac{\partial f^j}{\partial x_i},$$

where $(x_1, \ldots, x_{2n})$ denote real coordinates in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$.

With this perspective we can use the theory from Chapter 8 of [GT83]. Note that $u_{\xi\bar{\xi}}$ is thus a subsolution of the equation

$$Lv = g + \sum_j D_j f^j$$

where $g \equiv -C_1$, $D_j = \sum_i c_{i,j}\frac{\partial}{\partial x_i}$, f^j is bounded since $f \in C^{0,1}$, and

$$Lv = \sum_{i,j}\left(a^{i\bar{j}}v_i\right)_{\bar{j}}$$

The hypothesis on u and Lemma B.4 make the operator L uniformly elliptic (in the real sense). The weak Harnack inequality (see [GT83] Theorem 8.18) gives us:

$$r^{-2n}\int_{B_r}\left(\sup_{B_{4r}} u_{\xi\bar{\xi}} - u_{\xi\bar{\xi}}\right) \leqslant C_3\left(\sup_{B_{4r}} u_{\xi\bar{\xi}} - \sup_{B_r} u_{\xi\bar{\xi}} + r\right), \tag{6.2}$$

where $B_{4r} = B(z_0, 4r) \subset \Omega$ and $z_0 \in \Omega'$.

Set $U := \left(u_{i,\bar{j}}\right)$. For $x, y \in B_{4r} \subset \Omega$, we obtain

$$a^{i\bar{j}}(y)u_{i\bar{j}}(x) = f(y)u^{i\bar{j}}(y)u_{i\bar{j}}(x) = f(y)\operatorname{tr}\left(U(y)^{-1}U(x)\right).$$

In particular,

$$a^{i\bar{j}}(y)u_{i\bar{j}}(y) = nf(y).$$

On the other hand, $\det\left(f(y)^{1/n}U(y)^{-1}\right) = 1$, then with Lemma B.2 we get the inequality

$$a^{i\bar{j}}(y)u_{i\bar{j}}(x) = f(y)^{1-1/n}\operatorname{tr}\left(f(y)^{1/n}U(y)^{-1}U(x)\right) \geqslant nf(y)^{1-1/n}f(x)^{1/n}.$$

With the the expression above we get, for $x, y \in B_{4r}$,

$$\begin{aligned} a^{i\bar{j}}(y)\left(u_{i\bar{j}}(y) - u_{i\bar{j}}(x)\right) &\leqslant nf(y) - nf(y)^{1-1/n}f(x)^{1/n} \\ &= nf(y)^{1-1/n}\left(f(y)^{1/n} - f(x)^{1/n}\right) \\ &\leqslant C_4|x-y| \end{aligned} \tag{6.3}$$

where $C_4 > 0$ depends on $\sup f$ and the Lipschitz constant of $f^{1/n}$.

We now wish to combine Ineq 6.2 with Ineq 6.3. Notice that, the eigenvalues of $\left(a^{i\bar{j}}(y)\right)$ lie in $[\lambda, \Lambda]$, where $\lambda, \Lambda > 0$ are under control. By Lemma B.1, we can find unit vectors $\xi_1, \ldots, \xi_N \in \mathbb{C}^n$ such that for $x, y \in \Omega$

$$a^{i\bar{j}}(y)\left(u_{i\bar{j}}(y) - u_{i\bar{j}}(x)\right) = \sum_{k=1}^{N}\beta_k(y)\left(u_{\xi_k\bar{\xi}_k}(y) - u_{\xi_k\bar{\xi}_k}(x)\right)$$

where $\beta_k(y) \in [\lambda_*, \Lambda_*]$ and $\lambda_*, \Lambda_* > 0$ are under control. And hence we can control this types of amplitudes. Now set

$$M_{k,r} := \sup_{B_r} u_{\xi_k}\bar{\xi}_k, \quad m_{k,r} := \inf_{B_r} u_{\xi_k\bar{\xi}_k}$$

and

$$\eta(r) := \sum_{k=1}^{N} (M_{k,r} - m_{k,r})$$

We need to show that $\eta(r) \leqslant Cr^\alpha$. Since $\xi_1, \ldots, \xi_N$ can be chosen so that they contain the coordinate vectors, it will then follow that$||\Delta u||_{C^\alpha(\Omega')}$ is under control. By the Schauder estimates for the Poisson equation see Theorem 4.6 in [GT83] that $\left||D^2 u\right||_{C^\alpha(\Omega')}$ is also under control. To establish the Hölder condition $\eta(r) \leqslant Cr^\alpha$ it suffices to show that

$$\eta(r) \leqslant \delta\eta(4r) + r, \quad 0 < r < r_0 \tag{6.4}$$

where $\delta \in (0,1)$ and $r_0 > 0$ are under control (see Lemma 8.23 [GT83]).

It follows from Ineq 6.3 that

$$\sum_{k=1}^{N} \beta_k(y) \left(u_{\xi_k\bar{\xi}_k}(y) - u_{\xi_k\bar{\xi}_k}(x)\right) \leqslant C_4|x-y| \tag{6.5}$$

Summing 6.2 over $l \neq k$, where k is fixed, we obtain

$$r^{-2n} \int_{B_r} \sum_{l\neq k} \left(M_{l,4r} - u_{\xi_l\bar{\xi}_l}\right) \leqslant C_3(\eta(4r) - \eta(r) + r). \tag{6.6}$$

By 6.5, for $x \in B_{4r}, y \in B_r$ we have

$$\begin{aligned}\beta_k(y)\left(u_{\xi_k\bar{\xi}_k}(y)\right. & \left. -u_{\xi_k\bar{\xi}_k}(x)\right) \\ & \leqslant C_4|x-y| + \sum_{l\neq k} \beta_l(y)\left(u_{\xi_l\bar{\xi}_l}(x) - u_{\xi_l\bar{\xi}_l}(y)\right) \\ & \leqslant C_5 r + \Lambda^* \sum_{l\neq k}\left(M_{l,4r} - u_{\xi_l\bar{\xi}_l}(y)\right).\end{aligned}$$

The inequality above holds for all x, y in their proper domains. Hence on the left hand side x is a free variable. Thus for all $y \in B_r$,

$$u_{\xi_k\bar{\xi}_k}(y) - m_{k,4r} \leqslant \frac{1}{\lambda^*}\left(C_5 r + \Lambda^* \sum_{l\neq k}\left(M_{l,4r} - u_{\xi_l\bar{\xi}_l}(y)\right)\right)$$

and 6.6 gives

$$r^{-2n}\int_{B_r}\left(u_{\xi_k\bar{\xi}_k} - m_{k,4r}\right) \leqslant C_6(\eta(4r) - \eta(r) + r).$$

This together with 6.2 implies $\eta(r) \leqslant C_7(\eta(4r) - \eta(r) + r)$. Then the desired Inequality 6.4 follows.

□

Now to complete the proof of the results necessary for the proof of Theorem 2.8 we prove Theorem 6.2. We prove such theorem by a bootstraping argument using the classical Schauder Theory. Now that the $C^{2,\alpha} - estimates$ are available the argument should follow from a direct application.

The result we'll prove precisely is:

Theorem 6.2. *Assume* $(\omega + dd^c\varphi)^n = f\omega^n$*, where* $f > 0$ *and* φ *is* ω *plurisubharmonic and* $\mathcal{C}^{2,\alpha}$*-smooth. Then*

$$\varphi \in C^{2,\alpha}, f \in C^{k,\alpha} \Longrightarrow \varphi \in C^{k+2,\alpha}$$

and

$$||\varphi||_{k+2,\alpha} \leqslant C$$

where $C > 0$ *depends only on upper bounds for* $||\varphi||_{2,\alpha}, ||f||_{k,\alpha}$.

Proof. To apply Schauder Theory for the operator

$$F\left(D^2 u\right) = \det\left(u_{i\bar{j}}\right) \tag{6.7}$$

we need to have that it is Elliptic then by Lemma B.4 we get:[20]

The operator F is elliptic (in the real sense) for smooth strongly psh functions. Even *uniformily elliptic* when a $C^{2,\alpha}$ estimate is available. That is

$$|\xi|^2/C \leqslant \sum_{p,q=1}^{2n} \partial F/\partial u_{pq}\xi_p\xi_q \leqslant C|\xi|^2, \quad \xi \in \mathbb{C}^n = \mathbb{R}^{2n}$$

for some uniform constant C.

By Theorem 6.1 we have uniformly ellipticity of F and thus we apply the standard elliptic theory[21] to the equation

$$F\left(D^2 u\right) = f$$

For a fixed unit vector ξ and small $h > 0$ we consider

$$u^h(x) = \frac{u(x + h\xi) - u(x)}{h}$$

and

$$a_h^{pq} = \int_0^1 \frac{\partial F}{\partial u_{pq}}\left(tD^2u(x + h\xi) + (1-t)D^2u(x)\right) dt$$

Thus

$$a_h^{pq}(x)u_{pq}^h(x) = \frac{1}{h}\int_0^1 \frac{d}{dt}F\left(tD^2u(x + h\xi) + (1-t)D^2u(x)\right) dt = f^h(x)$$

Schauder theory for linear elliptic equations with variable coefficients yields the a priori estimates

$$u \in C^{2,\alpha} \Longrightarrow a_h^{pq} \in C^{0,\alpha} \overset{\text{Schauder}}{\Longrightarrow} u^h \in C^{2,\alpha}$$

Since these estimates are uniform in h we infer $u \in C^{3,\alpha}$. Now we do the *bootstraping argument.* We apply the Schauder estimates again recursively and then we have

$$u \in C^{3,\alpha} \Longrightarrow a_h^{pq} \in C^{1,\alpha} \overset{\text{Schauder}}{\Longrightarrow} u^h \in C^{3,\alpha} \Longrightarrow u \in C^{4,\alpha} \Longrightarrow \dots$$

With this we can achieve as much regularity as f can allow as it's regularity is the only barrier for the recursive process to continue. This completes the proof of Theorem 6.2

□

To actually get the smoothness we want it is just a matter of applying the above result/argument for arbitrary $k \in \mathbb{N}$ as in the case of Theorem 2.8 the f is C^∞.

[20] For more details regarding this and other complex operators seen as uniformly elliptic in the real sense refer to [Blo99].

[21] For a more detailed and general description of the procedure then the one on this notes see Lemma 17.16 [GT83].

Appendix A. Where do we go now?

Here in this appendix I'll present very briefly some problems that go beyond the scope of these notes for the interested reader.

A.1. **Calabi-Yau-Gauduchon Theorem.** The Calabi-Yau-Gauduchon (shortened by CYG) Theorem, posed as a conjecture by Gauduchon in [Gau84] on IV.5, is canonical type problem related to *Gauduchon Metrics* (Definition 2.1). This problem regards finding a Gauduchon metric whose volume form is prescribed. This problem can be seen as a natural generalization of the Hermitian Calabi-Yau Theorem 2.7. However, in dimension $n \geqslant 3$ the dd^c perturbation of a Gauduchon metric is not necessarily Gauduchon, then Theorem 2.7 does not help directly to prove the CYG Theorem.

This Theorem was proved by Tosatti-Weinkove-Szekelyhidi [STW17] and also showed that one can further prescribe the Aeppli class ($H_A^{n-1,n-1}(X)$) of the Gauduchon metric.

Now we present the two equivalent formulations of the conjecture which give us the interpretations above:

Conjecture A.1. *Let M be a compact complex manifold and η be a closed real $(1,1)$ form on M with $[\eta] = c_1^{\mathrm{BC}}(M) \in H^{1,1}_{\mathrm{BC}}(M,\mathbb{R})$. Then there is a Gauduchon metric ω on M with*

$$\mathrm{Ric}(\omega) = \eta$$

Conjecture A.2. *Let M be a compact complex manifold and σ be a smooth positive volume form. Then, there is a Gauduchon metric ω on M with*

$$\omega^n = \sigma$$

A.2. **Singular setting.** The extension of result of Kähler Geometry to mildly singular varieties has been a very active field of research on the past years. (see [GZ17] and the references therein for the pluripotential aspect of this story.) On the Hermitian setting some development has been made such as the definition of a *Singular Gauduchon Metric* for some types of sinularities by Chug-Ming Pan [Pan22]. Also the extension of Pluripotential techniques such as in [KN19, GL22, GL21] has allowed some progress in this endeavor.

One relevant still open question is whether one can prove that the volume of the dd^c perturbation of the metric can have infimum zero, i.e.

$$\inf\{\int_X (\omega + dd^c\varphi) | \varphi \in C^\infty(X,\mathbb{R}) \ \ with \ \ \omega + dd^c\varphi > 0\}$$

Metrics with such an infimum being non-zero are called *uniformly non-collapsing* and one such that there are no u bounded ω-psh such that $(\omega + dd^c u)^n \equiv 0$ is called *non-collapsing* in [GL22]. There has been shown that if a singular metric satisfies the condition (B) (see Definition2.2) then it is non-collapsing. In the same paper many examples of such metrics are given.

Appendix B. Inventory of Results

This appendix is a list of result and references we use throughout the proofs and decided to store here to keep the main text more organized.

Lemma B.1. *Let $0 < \lambda < \Lambda < \infty$ and let $S(\lambda,\Lambda)$ denote the set of Hermitian matrices whose eigenvalues lie in the interval $[\lambda,\Lambda]$.*

One can find unit vectors $\zeta_1,\dots,\zeta_N \in \mathbb{C}^n$ and $0 < \lambda_* < \Lambda_* < \infty$, depending only on n, λ, and Λ, such that every $A \in S(\lambda,\Lambda)$ can be written as

$$A = \sum_{k=1}^N \beta_k \zeta_k \otimes \bar{\zeta}_k, \qquad i.e.,\ a_{i\bar{j}} = \sum_k \beta_k \zeta_{ki} \bar{\zeta}_{kj}$$

where $\beta_k \in [\lambda_*, \Lambda_*]$, $k = 1,\dots,N$. The vectors $\zeta_1,\dots,\zeta_N$ can be chosen so that they contain a given orthonormal basis of $\mathbb{C}^n$.

(see [Siu87] *page 103 for a proof.)*

Lemma B.2. *Let $\mathcal{H}$ denote the set of all $n \times n$ Hermitian matrices and set*

$$\mathcal{H}_+ := \{A \in \mathcal{H} | A > 0\}$$

Then, for all $A \in \mathcal{H}_+$

$$(\det A)^{1/n} = \frac{1}{n} \inf\{tr(AB) | B \in \mathcal{H}_+, \det B = 1\}$$

(see Lemma 5.8 [GZ17] *for a proof.)*

Lemma B.3. *Let $\mathcal{H}$ denote the set of all $n \times n$ Hermitian matrices and set*

$$\mathcal{H}_+ := \{A \in \mathcal{H} | A > 0\}$$

Then the map

$$\mathcal{H}_+ \ni A \longmapsto (\det A)^{1/n} \in \mathbb{R}_+$$

is concave.

(This result can be deduced from the Lemma B.2 above.)

Lemma B.4. *One has for the complex Monge-Ampère operator 6.7*

$$\lambda_{\min}(\partial F / \partial u_{pq}) = \frac{\det(u_{i\bar{j}})}{4\lambda_{\max}(u_{i\bar{j}})}, \quad \lambda_{\max}(\partial F / \partial u_{pq}) = \frac{\det(u_{i\bar{j}})}{4\lambda_{\min}(u_{i\bar{j}})}.$$

(see [Blo99] *for more details.)*

Lemma B.5. *Let (X, ω) Hermitian manifold. Take $\alpha \in \Omega^{(p-1,p-1)}(X)$ and $\beta \in \Omega^{(n-p,n-p)}(X)$. Then:*

$$\int_X dd^c\alpha \wedge \beta = \int_X \alpha \wedge dd^c\beta.$$

Proof.

$$\begin{aligned}
\int_X dd^c\alpha \wedge \beta &= c\int_X \partial\bar{\partial}\alpha \wedge \beta \quad (Definition) \\
&= c\int_X d(\bar{\partial}\alpha) \wedge \beta \quad (\bar{\partial}^2 = 0) \\
&= c\int_X \bar{\partial}\alpha \wedge d\beta \quad (Stokes) \\
&= c\int_X \bar{\partial}\alpha \wedge \partial\beta \quad (Dimension) \\
&= c\int_X d\alpha \wedge \partial\beta \quad (Dimension) \\
&= -c\int_X \alpha \wedge d(\partial\beta) \quad (Stokes) \\
&= -c\int_X \alpha \wedge \bar{\partial}\partial\beta \quad (\partial^2 = 0) \\
&= c\int_X \alpha \wedge \partial\bar{\partial}\beta \quad (\partial\bar{\partial} = -\bar{\partial}\partial) \\
&= \int_X \alpha \wedge dd^c\beta \quad (Definition)
\end{aligned}$$

□

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