

# SINGULAR GAUDUCHON CONJECTURE

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ABSTRACT. In 1984 Gauduchon conjectured that one can find Gauduchon metrics with prescribed Ricci curvature on every compact complex manifolds. This conjecture was settled by Székelyhidi-Tosatti-Weinkove [TW17, TW19, STW17] via the study of a Monge-Ampère equation for  $(n - 1)$ -plurisubharmonic functions with a gradient term. In this paper we study a singular version of this conjecture. We obtain a  $C^0$ -estimate for this problem, without gradient terms, on smoothable hermitian varieties by adapting a technique from [GL25b]. We also prove smoothness of the solutions on holomorphic Kähler families, generalizing [TW17].

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## INTRODUCTION

Yau's solution of the Calabi conjecture [Yau78] prescribes the Ricci form of a Kähler metric in a fixed Kähler class. In the hermitian setting, Gauduchon [Gau84, IV.5] proposed the analogous problem for Gauduchon metrics, i.e., hermitian metrics  $\omega_G$  satisfying  $\partial\bar{\partial}\omega_G^{n-1} = 0$ . This problem, the Gauduchon conjecture, was settled by Székelyhidi, Tosatti and Weinkove [TW17, TW19, STW17]. In this article we study a singular version of it.

Gauduchon metrics are ubiquitous objects in hermitian geometry since they exist in the conformal class of any hermitian metric [Gau77a]. Recently, Barbaro and Otiman [BO25] investigated Calabi-Yau-Gauduchon metrics (Gauduchon + Chern-Ricci flat) on locally conformally Kähler manifolds. The Gauduchon cone, related to the pseudo-effective cone [Lam99], also motivates a finer study of such metrics, see [Pop15, PU18, OV25b, OV25a].

Degenerate PDEs on singular spaces are closely connected to the Minimal Model Program and to moduli theory. Notable progress includes the singular Kähler version of Yau's theorem by Eyssidieux, Guedj and Zeriahi [EGZ09] and the construction of Gauduchon metrics on singular hermitian varieties by Pan [Pan22, Pan25]. An important open problem in the singular setting is Reid's fantasy [Rei87], which conjectures that Calabi-Yau threefolds form a connected moduli space via conifold transitions introduced by Clemens and Friedman [Cle83, Fri86]; these transitions consist of smoothings and bimeromorphic transformations of non-Kähler Calabi-Yau varieties. These constructions are believed to be related

to Hull-Strominger systems [Hul86, Str86]. For these systems one needs to study a Calabi-Yau semi-Kähler metric (also known as balanced metrics), defined by  $d\omega_B^{n-1} = 0$  [Gau77b]; these metrics are closely related to the Calabi-Yau-Gauduchon equation considered here, see [FWW10, FWW15, TW17, GS25, Chu12, FLY12, CPY24].

The Calabi-Yau-Gauduchon problem on a hermitian manifold  $(M, \beta)$  equipped with a Gauduchon metric  $\beta$  reduces to finding a hermitian metric  $\alpha$  satisfying the following properties (see [STW17, Theorem 1.3]):

- (1)  $\alpha$  is Gauduchon, i.e.,  $\partial\bar{\partial}\alpha^{n-1} = 0$ .
- (2)  $[\alpha^{n-1}] = [\beta^{n-1}]$  in Aeppli cohomology  $H_A^{n-1, n-1}(M)$ .
- (3)  $Ric^{BC}(\alpha) = \eta$ , for a fixed closed real  $(1, 1)$ -form  $\eta \in c_1^{BC}(M)$ .

Recall that the  $(n-1, n-1)$ -Aeppli cohomology group is defined as:

$$H_A^{n-1, n-1}(M) = \frac{\{\partial\bar{\partial}\text{-closed real } (n-1, n-1)\text{-currents}\}}{\{\partial\gamma + \bar{\partial}\bar{\gamma} \mid \gamma \text{ } (n-2, n-1)\text{-current}\}}.$$

By [Pan22, Theorem A], for any compact smoothable hermitian variety  $X$  — that is, a variety approximable by a family of smooth manifolds as in Setting (SF) — there exists a singular Gauduchon metric: a hermitian metric  $\beta$  that is smooth and Gauduchon on the regular part  $X^{\text{reg}}$  and bounded on  $X$ . Such a metric is obtained as a limit of the metrics on nearby smooth fibers in the smoothing family. Moreover, by [Pan22, Theorem B],  $\beta^{n-1}$  extends to an  $(n-1, n-1)$  pluriclosed current  $T_G$  (i.e.,  $\partial\bar{\partial}T_G = 0$ ) which defines an Aeppli class; see [Pan25] for a generalization to compact hermitian varieties. Hence the first two conditions above admit singular analogues. Prescribing the Bott-Chern Ricci form on a singular variety is equivalent (see [GL23, Theorem 4.5]) to solving

$$\omega^n = \mu \tag{CY}$$

where  $\mu$  is the adapted volume (see Definition 1.20). Thus one can pose the singular Calabi-Yau-Gauduchon problem as follows.

**Problem CYG.** *Let  $X$  be a compact smoothable hermitian variety,  $\beta$  be a singular Gauduchon metric on  $X$  with Gauduchon class  $[\beta^{n-1}] \in H_A^{n-1, n-1}(X)$ , and  $\eta \in c_1^{BC}(X)$  be a closed real  $(1, 1)$ -form on  $X$ . Does there exist a singular metric  $\alpha$  on  $X$  such that*

- (1)  $\alpha$  is Gauduchon on  $X$ , i.e.,  $\partial\bar{\partial}\alpha^{n-1} = 0$  on  $X^{\text{reg}}$ .
- (2)  $[\alpha^{n-1}] = [\beta^{n-1}] \in H_A^{n-1, n-1}(X)$ .
- (3)  $Ric^{BC}(\alpha) = \eta$  on  $X^{\text{reg}}$ ?

**Remark.** *The expected solution should be more singular than the bounded Gauduchon metric of [Pan22, Definition 1.2]. Indeed, if a solution were quasi-isometric to a smooth metric, then  $X$  would be smooth by [GGZ24, Lemma 1.6]. Moreover, by [DNGG23, Lemma 4.4, eq (4.7)], the volume of the desired Gauduchon metric must lie in  $L^p$  relative to a smooth volume form on  $X$ , for some  $p > 1$ , whenever  $X$  has log-terminal singularities.*

Let us recall the notion of smoothing families:

**Setting (SF).** *Let  $\mathfrak{X}$  be an  $(n+1)$ -dimensional, irreducible, reduced, complex analytic space. Suppose that*

- $\pi : \mathfrak{X} \rightarrow \mathbb{D}$  is a proper, surjective, holomorphic map with connected fibers  $X_t := \pi^{-1}(t)$ ;
- $\pi$  is smooth over the punctured disk  $\mathbb{D}^*$ ;
- $X_0$  is an  $n$ -dimensional, normal compact complex analytic space;
- $X_0$  has only isolated singularities.

Let  $\omega, \beta$  be hermitian metrics on  $\mathfrak{X}$  in the sense of Definition 1.18. For each  $t \in \mathbb{D}$ , we denote by  $\omega_t$  the hermitian metric on fiber  $X_t$  obtained by restriction,  $\omega_t := \omega|_{X_t}$ .

Call  $X$  smoothable if it arises as such a central fiber  $X_0 = X$  for some family  $\mathfrak{X}$  as above.

In the smooth case, Problem CYG is solved in [STW17, Theorem 1.4] via the studying a complex Monge-Ampère equation for  $(n-1)$ -plurisubharmonic functions with a gradient term. This equation was discovered by Popovici [Pop15] and, independently, by Tossati and Weinkove [TW19] (see [FWW10, FWW15] for earlier related discussions). In this paper, we restrict ourself to the case without the gradient term, studied in [TW17]; that is:

For  $(\mathfrak{X}, \beta, \omega)$  a smoothing as in Setting (SF), fix  $p > 1$  and  $0 \leq f_t \in L^p(X_t, \omega_t^n)$  be a family of densities,  $1 \leq \rho_t \in L^\infty(X_t)$  be a family of functions with  $f_t, \rho_t \in C^\infty(X_t), \forall t \neq 0$ . Assume the families  $(f_t)_{t \in \mathbb{D}}$  and  $(\rho_t)_{t \in \mathbb{D}}$  satisfy: there exist constants  $c_f, C_f, G > 0$  such that  $\forall t \in \mathbb{D}$ ,

$$c_f \leq \int_{X_t} f_t^{\frac{1}{n}} \omega_t^n \quad \text{and} \quad \|f_t\|_{L^p(X_t, \omega_t^n)} \leq C_f, \quad (\text{DB})$$

$$\sup_{X_t} \rho_t \leq G \quad \text{and} \quad \inf_{X_t} \rho_t = 1. \quad (\text{GB})$$

We study the following equation with the conditions  $\sup_{X_t} \varphi_t = 0$  and  $\alpha_t > 0$ :

$$\alpha_t^n := \left( \rho_t \beta_t + \frac{(\Delta_{\omega_t} \varphi_t) \omega_t - dd_t^c \varphi_t}{n-1} \right)^n = c_t f_t \omega_t^n, \quad (\text{CYG})$$

where  $(\varphi_t, c_t) \in L^\infty(X_t) \cap \text{PSH}_{n-1}(X_t, \rho_t \beta_t, \omega_t) \times \mathbb{R}_{>0}$  is the unknown. The set  $\text{PSH}_{n-1}(X, \beta, \omega)$  of  $(n-1)$ -quasi plurisubharmonic functions on a space  $X$  with two hermitian metrics  $\beta, \omega$  can be defined (see Definition 1.13) as the set of quasi-subharmonic functions such that  $\alpha_\varphi := \beta + \frac{(\Delta_\omega \varphi) \omega - dd^c \varphi}{n-1} \geq 0$  in the sense of currents.

By [DNGG23, Lemma 4.4, eq (4.7)], the adapted volume of smoothings satisfy the integral bounds (DB) when the central fiber  $X_0$  has log-terminal singularities. By [Pan22, Theorem A], the Gauduchon factors of  $\beta_t$  satisfy the uniform bound (GB). Hence an  $L^\infty$ -bound for the problem above yields a bound for a simplified singular Calabi-Yau-Gauduchon problem.

We can state now our first result:

**Theorem A.** *Under Setting (SF) and the uniform hypotheses above (DB) (GB), let, for each  $t \in \mathbb{D}$ , the pair  $(\varphi_t, c_t) \in \text{PSH}_{n-1}(X_t, \beta_t, \omega_t) \cap L^\infty(X_t) \cap \mathbb{R}_{>0}$  be the solution to the complex  $(n-1)$ -Monge-Ampère equation (CYG). Then there exists a constant  $C > 0$  uniform, independent of  $t$ , such that  $\forall t \in \mathbb{D}_{1/2}$ ,*

$$c_t + c_t^{-1} + \|\varphi_t\|_{L^\infty} \leq C.$$

For the precise dependence of  $C$ , see Theorem 2.6.

The higher-order estimates obtained in [TW17] and [Pan22] are non-quantitative: the former relies on a blow-up analysis for the  $C^1$ -estimate while the latter uses a localization argument. Consequently, obtaining higher-order estimates in the presence of singularities for Equation (CYG) is more challenging. Taking this into account, we focus the remainder of the paper on families without singular fiber, i.e., deformations of the complex structure of a fixed complex manifold  $M$ . If  $\omega$  is Astheno-Kähler ( $\partial \bar{\partial} \omega^{n-2} = 0$ ), then a solution of Equation (CYG) provides a solution to Problem CYG, without condition (2). Also, the Astheno-Kähler condition is not stable under small deformations [FT09, Sfe23]; therefore we assume  $(M, \omega)$  is Kähler, which is stable [KS60] and the solution solves Problem CYG.

To emphasize that in the second part there are no singular fibers, we denote individual complex manifolds by  $M$  and holomorphic families by  $\mathcal{M}$ .

In this setting we have our second result:

**Theorem B.** *Let  $(\mathcal{M}, \beta, \omega)$  be a holomorphic family satisfying the [Setting \(SF\)](#), with central fiber  $M_0$  a smooth Kähler manifold and with  $\omega$  a Kähler metric. Let  $f \in C^\infty(\mathcal{M})$ , and write  $f_t := f|_{M_t} \in C^\infty(M_t)$ , for  $M_t := \pi^{-1}(t)$ . Then for each  $t \in \mathbb{D}_{\frac{1}{2}}$  there exists a unique solution  $(u_t, c_t) \in \text{PSH}_{n-1}(X, \beta, \omega) \cap C^\infty(M_t) \times \mathbb{R}_{>0}$  of*

$$\left( \beta_t + \frac{1}{n-1} \left( (\Delta_{\omega_t} u_t) \omega_t - dd^c u_t \right) \right)^n = c_t f_t \beta_t^n, \quad \sup_{M_t} u_t = 0$$

and for every  $k \in \mathbb{N}$

$$\|u_t\|_{C^k(M_t)} + c_t + c_t^{-1} \leq C$$

for a uniform  $C > 0$  independent of  $t$ . Hence,  $C$  depends on the complex structure of  $M$ .

The preceding result produces, on a general Kähler manifold  $M$ , non-Kähler Gauduchon (or semi-Kähler) metrics with prescribed Ricci curvature that vary smoothly with the complex structure, provided  $\beta$  is a non-Kähler Gauduchon (or semi-Kähler) metric: the  $C^\infty$ -bounds are uniform and independent of the complex structure. See [Section 1](#) or [\[TW17\]](#).

### Organization of the article:

- [Section 1](#) familiarizes the reader with  $(n-1)$ -plurisubharmonic functions in  $\mathbb{C}^n$  and on manifolds, and it introduces the singular setting.
- [Section 2](#) establishes two different uniform bounds for the normalizing constant and proves [Theorem A](#).
- [Sections 3](#) and [4](#) extend the  $C^2, C^1$  and higher-order estimates of [\[TW17\]](#) to holomorphic families and complete the proof of [Theorem B](#)

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## 1. QUASI $(n-1)$ -PLURISUBHARMONIC FUNCTIONS

We set  $d^c = \frac{i}{2}(\bar{\partial} - \partial)$  so that  $dd^c = i\partial\bar{\partial}$ ,  $\omega_{\mathbb{C}^n}$  the standard hermitian metric on  $\mathbb{C}^n$ . We fix the complex dimension  $n \geq 3$ , since for  $n = 2$  the notion of  $(n-1)$ -plurisubharmonic coincides with that of plurisubharmonic.

We define the complex Laplacian with respect to a hermitian metric  $\omega$  as:

$$\Delta_\omega g := \text{tr}_\omega(dd^c g) = n \frac{dd^c g \wedge \omega^{n-1}}{\omega^n}.$$

1.1.  **$(n-1)$ -plurisubharmonic functions.** Fix  $\Omega \subset \mathbb{C}^n$  a domain.

1.1.1. *Basic properties and examples.*

**Definition 1.1.** *We say that  $u \in C^2(\Omega)$  is a  $(n-1)$ -plurisubharmonic,  $(n-1)$ -psh for short, function on  $\Omega$  if the following  $(1,1)$ -form is semi-positive*

$$\widehat{dd^c u} := (\Delta u) \omega_{\mathbb{C}^n} - dd^c u \geq 0.$$

Note that  $\widehat{dd^c(\cdot)}$  is a linear operator. One has the following equivalent definitions which can be found throughout the literature, see [HL11, HL12, Ver10, Din22].

**Proposition 1.2.** *Let  $u \in C^2(\Omega)$ . Then  $u$  is  $(n-1)$ -plurisubharmonic on  $\Omega$  if and only if one of the following equivalent conditions is satisfied:*

- (1) *The eigenvalues  $\{\lambda_i\}_{i=1}^n$  of the complex hessian of  $u$ , satisfy  $\widehat{\lambda}_i := \sum_{k \neq i} \lambda_k \geq 0$ .*
- (2) *For any complex hyperplane of dimension  $n-1$ ,  $H \subset \mathbb{C}^n$ ,  $u|_{H \cap \Omega}$  is subharmonic.*
- (3) *The  $(n-1, n-1)$ -form  $dd^c u \wedge \omega_{\mathbb{C}^n}^{n-2}$  is (weakly) positive.*
- (4) *The  $(n-1, n-1)$ -form  $dd^c u \wedge \omega_{\mathbb{C}^n}^{n-2}$  is strongly positive.*

We refer the reader to [Dem12, Chapter III, Corollary 1.9] for (3)  $\iff$  (4). We highlight that (3)  $\iff$   $u$  is  $(n-1)$ -psh follows from the Hodge star computation:

$$\frac{1}{(n-1)!} * (dd^c u \wedge \omega_{\mathbb{C}^n}^{n-2}) = \frac{1}{n-1} ((\Delta u) \omega_{\mathbb{C}^n} - dd^c u) = \frac{1}{n-1} \widehat{dd^c u}.$$

We follow the conventions for the Hodge star operator of [TW17, Section 2]; that is, for  $p, q \in \{1, \dots, n\}$  the Hodge star operator  $*$  (induced by  $\omega_{\mathbb{C}^n}$  in this subsection) maps a  $(p, q)$ -form  $\psi$  into a  $(n-p, n-q)$ -form  $*\psi$  such that for any  $(p, q)$ -form  $\varphi$ :

$$\varphi \wedge *\bar{\psi} = \langle \varphi, \psi \rangle_{\omega_{\mathbb{C}^n}} \frac{\omega_{\mathbb{C}^n}^n}{n!}, \quad *\bar{\psi} = \overline{*\psi}, \quad **\psi = (-1)^{p+q}\psi, \quad (1.1)$$

From Proposition 1.2 one notices that  $u$  is subharmonic, hence we define  $m$ -subharmonic functions to compare these notions of positivity.

**Definition 1.3.** *Let  $1 \leq m \leq n$ . We say  $u \in C^2(\Omega)$  is  $m$ -subharmonic on  $\Omega$  if*

$$(dd^c u)^k \wedge \omega_{\mathbb{C}^n}^{n-k} \geq 0, \quad \forall 1 \leq k \leq m.$$

We denote  $\text{PSH}_{n-1}(\Omega)$  the set of  $(n-1)$ -plurisubharmonic ( $(n-1)$ -psh for short) functions in  $\Omega$ ,  $\text{SH}_m(\Omega)$  the set of  $m$ -subharmonic functions and  $\text{PSH}(\Omega) = \text{SH}_n(\Omega)$  the plurisubharmonic functions in  $\Omega$ . All these sets exclude the function  $u \equiv -\infty$ . We introduce non-regular  $(n-1)$ -psh functions:

**Definition 1.4.** *Let  $u \in \text{SH}_1(\Omega)$  be a subharmonic function on  $\Omega$ . Then  $u \in \text{PSH}_{n-1}(\Omega)$  if:*

$$\widehat{dd^c u} := (\Delta u) \omega_{\mathbb{C}^n} - dd^c u \geq 0$$

*as a real current of bidegree  $(1, 1)$ .*

**Proposition 1.5.** *Let  $u, v \in \text{PSH}_{n-1}(\Omega)$  and  $\lambda > 0$ . Then,*

- (i)  $u + v, \lambda u, \max\{u, v\} \in \text{PSH}_{n-1}(\Omega)$ .
- (ii) *If  $(w_i)_{i=1}^{+\infty} \subset \text{PSH}_{n-1}(\Omega)$  and  $w_i \searrow w \not\equiv -\infty$  pointwise on  $\Omega$ , then  $w \in \text{PSH}_{n-1}(\Omega)$ .*
- (iii) *If  $u$  is continuous and  $\chi_\varepsilon$  is a family of mollifiers, then their convolution  $u_\varepsilon = u \star \chi_\varepsilon$  is smooth  $(n-1)$ -psh function and converges to  $u$  locally uniformly.*
- (iv)  $\text{PSH}(\Omega) \subset \text{SH}_2(\Omega) \subset \text{PSH}_{n-1}(\Omega) \subset \text{SH}_1(\Omega)$ .

The proofs for (i)-(iii) above are analogous to the classic ones for subharmonic functions regardless of the definition chosen. The inclusion  $\text{SH}_2(\Omega) \subset \text{PSH}_{n-1}(\Omega)$  holds since the eigenvalues  $\{\lambda_i\}_{i=1}^n$  of a smooth function in  $\text{SH}_2(\Omega)$  satisfy  $(\sum_{i=1}^n \lambda_i)^2 - \sum_{i=1}^n \lambda_i^2 \geq 0 \implies \sum_{i=1}^n \lambda_i \geq \lambda_j, \forall j \in \{1, \dots, n\}$  since  $\sum_{i=1}^n \lambda_i \geq 0$ .

**Example 1.6.** (1)  $g_\varepsilon(z) = -\frac{1}{(|z|^2 + \varepsilon^2)^{(n-2)}} \in C^\infty(\mathbb{C}^n) \cap \text{PSH}_{n-1}(\mathbb{C}^n)$  for  $\varepsilon > 0$ . Notice that

$$g_\varepsilon \searrow G_{n-1}(z) = -\frac{1}{|z|^{2(n-2)}} \in \text{PSH}_{n-1}(\mathbb{C}^n) \text{ and } G_{n-1}(0) = -\infty. \text{ Recall that } n \geq 3.^1$$

<sup>1</sup>One can also prove  $G_{n-1} \in \text{PSH}_{n-1}(\mathbb{C}^n)$  by [HL14, Theorem 9.2] as the origin is a removable singularity.

- (2)  $u : \mathbb{C}^3 \rightarrow \mathbb{R}$ ,  $u(z_1, z_2, z_3) = -|z_1|^2 + |z_2|^2 + |z_3|^2$  belongs to  $\text{PSH}_{n-1}(\mathbb{C}^3) \setminus \text{SH}_2(\mathbb{C}^3)$ .  
(3)  $v : \mathbb{C}^3 \rightarrow \mathbb{R}$ ,  $v(z_1, z_2, z_3) = -\frac{3}{2}|z_1|^2 + |z_2|^2 + |z_3|^2$  belongs to  $\text{SH}_1(\mathbb{C}^3) \setminus \text{PSH}_{n-1}(\mathbb{C}^3)$ .

Now we define the complex  $(n-1)$ Monge-Ampère operator  $\widehat{MA}$  as

**Definition 1.7.** Given  $u \in C^2(\Omega) \cap \text{PSH}_{n-1}(\Omega)$ , we set

$$\widehat{MA}(u) := \det(\widehat{dd^c}u)$$

in terms of eigenvalues we have

$$\widehat{MA}(u) = \prod_{i=1}^n \widehat{\lambda}_i = \prod_{i=1}^n \left( \sum_{j \neq i} \lambda_j \right)$$

where  $\{\lambda_i\}_{i=1}^n$  are the eigenvalues of the complex hessian of  $u$ .

**Remark 1.8.** Let's stress that the operator  $\widehat{MA}$  does not satisfy some properties of the complex Monge-Ampère operator such as the comparison principle by [Din22, Theorem 4.1]. This makes the development of a pluripotential theory for this operator more delicate.

The notion of maximality for  $(n-1)$ -plurisubharmonic functions, crucial for the blow-up analysis that will be constructed in Section 4.1.

**Definition 1.9.** We say  $u \in \text{PSH}_{n-1}(\Omega)$  is maximal if for any relatively compact open set  $\Omega_1 \Subset \Omega$  and  $\forall v \in \text{PSH}_{n-1}(\Omega_2)$  such that  $\Omega_1 \Subset \Omega_2 \Subset \Omega$  and  $v \leq u$  on  $\partial\Omega_1$  then  $v \leq u$  on  $\Omega_1$ .

### 1.1.2. Radial Examples.

**Lemma 1.10.** Let  $G_{n-1} : \mathbb{C}^n \rightarrow \mathbb{R}$  as in Example 1.6 and  $\chi : \mathbb{R}_{\leq 0} \rightarrow \mathbb{R}$  smooth. Then  $u(z) = \chi(G_{n-1}(z)) \in \text{PSH}_{n-1}(\mathbb{C}^n)$  if and only if  $\chi$  is non-decreasing and convex.

*Proof.* A direct computation shows that the eigenvalues of the complex hessian of  $u$  are:  $\lambda_1 = (n-2)|z|^{-2(n-1)}[(n-2)|z|^{-2(n-2)}\chi'' - (n-2)\chi']$ ,  $\lambda_j = (n-2)|z|^{-2(n-1)}\chi'$ ,  $\forall j > 1$ , hence, the eigenvalues of  $\widehat{dd^c}u$  are:

$$\widehat{\lambda}_1 = (n-1)(n-2)|z|^{-2(n-1)}\chi', \quad \widehat{\lambda}_j = (n-2)^2|z|^{-2(2n-3)}\chi''; \quad \forall j > 1.$$

This concludes the proof.  $\square$

By the Lemma 1.10 above  $\widehat{MA}$  for radial functions takes the form:

$$\widehat{MA}(u) = (n-1)(n-2)^{2n-1}|z|^{-4(n-1)^2}(\chi' \circ G_{n-1})(\chi'' \circ G_{n-1})^{n-1}.$$

Example 1.11 below shows a gap between the integrability of  $\widehat{MA}(u)$  and the classical Monge-Ampère case for the  $L^1(\log L)^p$ -norm.

**Example 1.11.** Let  $\chi(t) = -\log(\log(\log(-t)))$ , computing near zero we obtain:

$$\chi'(t) = \frac{1}{(-t)\log(-t)\log(\log(-t))}, \quad \chi''(t) \sim -\frac{1}{(-t)^2\log(-t)\log(\log(-t))}.$$

Hence, the density  $f := \widehat{MA}(u)$ , for  $u(z) = \chi(G_{n-1}(z))$  satisfies near:

$$f \sim \frac{1}{|z|^{2n}(-\log(|z|))^n(\log(-\log(|z|)))^n}, \quad \text{and} \quad \log(f) \sim -\log(|z|),$$

then near zero one has

$$\int_0^{+\infty} f(\log(f))^p \sim \int_0^{+\infty} \frac{1}{|z|^{2n}(-\log|z|)^{n-p}(\log(-\log|z|))^n} dV_{\mathbb{C}^n}$$

which is finite if and only if  $p \leq n-1$ . However, for the Monge-Ampère operator it is known that the critical exponent is  $p = n$ , see [GL25b, Example 2.4].

**Remark 1.12.** A computation shows  $\left(\widehat{MA}(G_{n-1})\right)^{\frac{1}{n-1}} = \delta_0$ , the Dirac mass at the origin.

**1.2. Quasi  $(n-1)$ -plurisubharmonic functions on manifolds.** Throughout,  $(M, \beta, \omega)$  denotes a hermitian manifold with two hermitian metrics. Fix a constant  $C_{\beta, \omega} > 1$  such that

$$C_{\beta, \omega}^{-1} \beta \leq \omega \leq C_{\beta, \omega} \beta. \quad (\text{GC})$$

1.2.1. *Smooth case.*

**Definition 1.13.** Let  $u \in C^2(M)$ , we say  $u$  is a quasi  $(n-1)$ -plurisubharmonic function relative to  $(\beta, \omega)$ , if the following form is semi-positive

$$\alpha_u := \beta + \frac{(\Delta_\omega u)\omega - dd^c u}{n-1} \geq 0.$$

We let  $C^2(M) \cap \text{PSH}_{n-1}(M, \beta, \omega)$  denote the set of  $(C^2)$ -smooth quasi  $(n-1)$ -plurisubharmonic functions relative to  $(\beta, \omega)$ ,  $\text{PSH}_{n-1}(M, \omega)$  when  $\beta = \omega$ ,  $\text{SH}_1(M, \beta, \omega)$  the set of quasi-subharmonic functions relative to  $(\beta, \omega)$  and  $\text{PSH}(M, \omega)$  the set of quasi-plurisubharmonic functions relative to  $\omega$ . All of these sets exclude the function  $u \equiv -\infty$ .

Taking the trace of  $\alpha_u$  with respect to  $\omega$  we explain in Proposition 2.3 that  $u$  as above is quasi-subharmonic with respect to  $(\beta, \omega)$ . For basic properties of these functions we refer to [HL11, HL12, ADO23, AO22, TW17, Pop15].

The operator of interest, namely the complex  $(n-1)$ Monge-Ampère operator  $MA_{n-1}$ , is:

**Definition 1.14.** Given  $u \in C^2(M) \cap \text{PSH}_{n-1}(M, \beta, \omega)$ , we define  $A_u : T^{1,0}M \rightarrow T^{1,0}M$  the endomorphism defined by  $\alpha_u$  relative to  $\omega$ , i.e.,  $A_u := \omega^{-1} \cdot \alpha_u$ . And set

$$MA_{n-1}(\beta, \omega, u) := \det(A_u).$$

Notice that for any constant  $C > 0$ ,  $MA_{n-1}(\beta, C\omega, u) = C^{-n} MA_{n-1}(\beta, \omega, u)$ .

**Remark 1.15.** Let  $(M, \omega)$  be a Kähler manifold. If  $\Phi_u = \omega_0^{n-1} + dd^c u \wedge \omega^{n-2}$ , for a Gauduchon non-Kähler metric  $\omega_0$  on  $M$ , then  $\Phi_u^{\frac{1}{n-1}}$  is a Gauduchon non-Kähler metric<sup>2</sup> and  $[\omega_0^{n-1}] = [\Phi_u]$  in Aeppli cohomology for any  $u \in C^2(M) \cap \text{PSH}_{n-1}(M, \omega_0, \omega)$ . That is the case because  $\omega$  is Kähler and by the equivalence of definitions for  $(n-1)$ -psh functions on Proposition 1.2. By [TW17, Equation (2.2)]  $\frac{\det(\Phi_u)}{\det(\omega^{n-1})} = \frac{\alpha_u^n}{\omega^n}$ ; hence solving the  $(n-1)$ -Monge-Ampère equation for  $(1, 1)$ -forms is equivalent to solving it for  $(n-1, n-1)$ -forms which is equivalent to solving the Gauduchon conjecture in this setting.

1.2.2. *Non-smooth case.*

**Definition 1.16.** Let  $u \in \text{SH}_1(M, \beta, \omega)$  be a quasi-subharmonic function on  $M$ . We say  $u$  is a quasi  $(n-1)$ -plurisubharmonic function relative to  $(\beta, \omega)$  if:

$$\alpha_u := \beta + \frac{(\Delta_\omega u)\omega - dd^c u}{n-1} \geq 0$$

as a current of bidegree  $(1, 1)$ .

This definition is equivalent to those in [HL12]. Denote by  $\text{PSH}_{n-1}(M, \beta, \omega)$  be the set of all, quasi  $(n-1)$ -plurisubharmonic functions relative to  $(\beta, \omega)$  that are not identically  $-\infty$ .

**Remark 1.17.** We also refer the reader to [CX25] for a viscosity approach to the weak theory of  $(n-1)$ -plurisubharmonicity and  $(n-1)$ Monge-Ampère.

<sup>2</sup>See [Mic82, Lemma 4.8] for the existence and uniqueness of the  $(n-1)$ th root.

**1.3. Analysis on complex analytic spaces.** Let  $X$  be a compact normal complex analytic space of pure dimension  $n \geq 1$ . Denote by  $X^{\text{reg}}$  the complex manifold of regular points of  $X$  and  $X^{\text{sing}} := X \setminus X^{\text{reg}}$  the set of singular points, which is an analytic subset of  $X$  with complex codimension  $\geq 2$ , since  $X$  is normal.

By definition, for each point  $x_0 \in X$ , there exists a neighborhood  $U$  of  $x_0$  and a local embedding  $j : U \hookrightarrow \mathbb{C}^N$  onto an analytic subset of  $\mathbb{C}^N$  for some  $N \geq 1$ . These local embeddings allow us to define the spaces of smooth forms of given degree on  $X$  as smooth forms on  $X^{\text{reg}}$  that are locally on  $X$  restrictions of an ambient form on  $\mathbb{C}^N$ . Other differential notions and operators, such as holomorphic, plurisubharmonic and  $(n-1)$ -plurisubharmonic functions, can also be defined in this way and currents are defined by duality (see [Dem85]).

**Definition 1.18.** *A hermitian metric  $\omega$  on  $X$  is a hermitian metric  $\omega$  on  $X^{\text{reg}}$  such that given any local embedding  $X \xrightarrow{\text{loc}} \mathbb{C}^N$ ,  $\omega$  extends smoothly to a hermitian metric on  $\mathbb{C}^N$ .*

Following the definition introduced in [Pan22, Definition 1.2] we set:

**Definition 1.19.** *A hermitian metric  $\omega_G$  on a variety  $X$  is:*

- *Gauduchon if it satisfies  $dd^c(\omega_G^{n-1}) = 0$  on  $X^{\text{reg}}$ ;*
- *bounded Gauduchon if there exists a smooth hermitian metric  $\omega$  on  $X$  and a positive function  $\rho \in L^\infty(X) \cap C^\infty(X^{\text{reg}})$  such that  $\omega_G = \rho\omega$  and  $dd^c(\rho\omega)^{n-1} = 0$  on  $X^{\text{reg}}$ .<sup>3</sup>*

1.3.1. *Adapted volume and canonical singularities.*

**Definition 1.20.** *Suppose that  $X$  is  $\mathbb{Q}$ -Gorenstein, that is  $X$  is normal and  $K_X$  is  $r$ -Cartier for some  $r \in \mathbb{N}$ , i.e.,  $rK_X$  is a rank 1 locally free sheaf. If  $\sigma$  is a trivialization of  $rK_X$  over  $X^{\text{reg}}$  and  $h^r$  a smooth metric on  $rK_X$  we consider the adapted volume form:*

$$\mu_h := \left( \frac{i^{rn^2} \sigma \wedge \bar{\sigma}}{|\sigma|_{h^r}^2} \right)^{\frac{1}{r}}$$

It is independent of the choice of trivialization  $\sigma$ . The term "variety" will always refer to a  $\mathbb{Q}$ -Gorenstein complex analytic space throughout this paper.

**Definition 1.21.** *A hermitian variety  $X$  has log-terminal singularities if for any log-resolution of singularities  $p : \tilde{X} \rightarrow X$ , for any local generator  $\tau$  of  $rK_X$ , the pole along any component  $E$  of  $\text{exc}(p)$  of the meromorphic  $r$ -canonical form  $p^*\alpha$  on  $X$  is of order  $\leq r-1$ .*

Now we define the (Bott-Chern) Ricci curvature in this context

**Definition 1.22.** *Let  $X$  be a compact hermitian variety,  $\mu$  the reference adapted volume relative to  $h^r \equiv 0$  on local trivializations of  $rK_X$ , we define the (Bott-Chern) Ricci curvature current of a metric  $\omega$ , for  $\mu = f\omega^n$ , as*

$$\text{Ric}^{BC}(\omega) := -dd^c \log \left( \frac{\omega^n}{\mu} \right) = -dd^c \log(f)$$

Let us stress that, by [GGZ24, Lemma 1.6], the Ricci curvature is not a smooth form unless  $X$  is smooth, see [EGZ09, Sections 5, 6 and 7] for more details.

<sup>3</sup>Our convention for the Gauduchon factor differs from Pan's by the omission of the  $\frac{1}{n-1}$  power.

1.3.2. *Smoothable varieties.* A smoothable variety is a hermitian variety that appears as the central fiber of some family satisfying [Setting \(SF\)](#). Set  $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ ,  $\mathbb{D}_r^* := \mathbb{D}_r \setminus 0$  and  $\mathbb{D} := \mathbb{D}_1$ . The volumes of the fibers  $(X_t, \omega_t)$  are uniformly controlled: there exists a uniform constant  $C_V \geq 1$  such that  $\forall t \in \overline{\mathbb{D}}_{1/2}$

$$C_V^{-1} \leq \text{Vol}_{\omega_t}(X_t) \leq C_V. \quad (\text{VB})$$

For non-Kähler examples of smoothable varieties one can look at [\[LT96, Theorem 1\]](#) and for examples of non-smoothable singularities one can look at [\[Har74, Section 0.1\]](#).

Recall that by [\[DNGG23, Lemma 4.4, equation \(4.7\)\]](#) the adapted volume of smoothings satisfy these integral bounds [\(DB\)](#) when the central fiber  $X_0$  has log-terminal singularities and the Gauduchon factors of  $\beta_t$  satisfy, by [\[Pan22, Theorem A\]](#), the uniform bound [\(GB\)](#).

1.4. **Useful Results.** The following result is a simplified version of [\[Pan23, Proposition 3.3\]](#) a global uniform Skoda integrability result. It will be relevant for the bound on [Proposition 2.5](#).

**Proposition 1.23.** *If  $\pi : (\mathcal{X}, \omega) \rightarrow \mathbb{D}$  is a family satisfying [Setting \(SF\)](#). Then there exists constants  $\alpha, A_\alpha$  such that for all  $t \in \overline{\mathbb{D}}_{1/2}$  and for all  $u_t \in \text{PSH}(X_t, \omega_t)$  with  $\sup_{X_t} u_t = 0$ ,*

$$\int_{X_t} e^{-\alpha u_t} \omega_t^n \leq A_\alpha. \quad (\text{SB})$$

We state next [\[TW17, Theorem 5.2\]](#) the Liouville theorem for  $(n-1)$ -plurisubharmonic functions. It will be crucial for the blow-up argument in [Theorem 4.1](#).

**Theorem 1.24.** *If  $u : \mathbb{C}^n \rightarrow \mathbb{R}$  is an  $(n-1)$ -plurisubharmonic function in  $\mathbb{C}^n$  which is Lipschitz continous, maximal and satisfies*

$$\sup_{\mathbb{C}^n} (|u| + |\nabla u|) < +\infty$$

*then  $u$  is a constant.*

## 2. $L^\infty$ A PRIORI ESTIMATE ON SMOOTHABLE HERMITIAN VARIETIES

2.1. **Comparison results and useful facts.** Now we relate the different notions of positivity and prove a few useful facts about  $(n-1)$ -plurisubharmonic functions.

**Proposition 2.1.** *For  $(M, \beta, \omega)$  hermitian manifold with two hermitian metrics and a constant  $C_{\beta, \omega} > 0$  such that  $C_{\beta, \omega}^{-1} \omega \leq \beta$ . Then  $\text{PSH}_{n-1}(M, C_{\beta, \omega}^{-1} \omega) \subset \text{PSH}_{n-1}(M, \beta, \omega)$ .*

*Proof.* We have the following computation:

$$\begin{aligned} \alpha_u &= \beta + \frac{1}{n-1} ((\Delta_\omega u) \omega - dd^c u) = \beta + C_{\beta, \omega}^{-1} \omega - C_{\beta, \omega}^{-1} \omega + \frac{1}{n-1} ((\Delta_\omega u) \omega - dd^c u), \\ &= (\beta - C_{\beta, \omega}^{-1} \omega) + (C_{\beta, \omega}^{-1} \omega + \frac{1}{n-1} ((\Delta_{C_{\beta, \omega}^{-1} \omega} u) C_{\beta, \omega}^{-1} \omega - dd^c u)), \\ &\geq C_{\beta, \omega}^{-1} \omega + \frac{1}{n-1} ((\Delta_{C_{\beta, \omega}^{-1} \omega} u) C_{\beta, \omega}^{-1} \omega - dd^c u). \end{aligned}$$

As we wanted to prove. □

When carrying out computations in coordinates, we will frequently use, at a fixed point  $x \in M$  a chart  $U_x$  such that  $\omega_{i\bar{j}}(x) = \delta_{i\bar{j}}$ ,  $\beta(x) = \gamma_i \delta_{i\bar{j}}$  and  $dd^c u(x) = \lambda_i \delta_{i\bar{j}} + \text{O.D.T.}$  for some smooth function  $u$ , where O.D.T. stands for "off-diagonal terms" (similarly to [\[TW17\]](#)) containing  $dz^k \wedge d\bar{z}^l$  for  $k \neq l$ . However those terms won't affect our computations.

Moreover, the condition of  $u \in \text{PSH}_{n-1}(M, \beta, \omega)$  can be interpreted as:

$$\widehat{\lambda}_i := \sum_{\substack{j=1 \\ j \neq i}}^n \lambda_j \geq -(n-1)\gamma_i, \quad \forall i \in \{1, \dots, n\}.$$

We first prove a domination principle for  $\text{PSH}_{n-1}(M, \beta, \omega)$  as in [GL25a, Corollary 2.3]. Our proof is simpler because we only need to deal with smooth functions:

**Lemma 2.2.** *Let  $u, v \in \text{PSH}_{n-1}(M, \beta, \omega) \cap C^\infty(M)$  such that  $\alpha_u^n \leq c\alpha_v^n$  on  $\{u < v\}$  for some  $c \in [0, 1)$ . Then  $u \geq v$ .*

*Proof.* Assume by contradiction that  $\{u < v\}$  is non-empty and fix  $x_0 \in M$  such that
$$(u - v)(x_0) = \min_M(u - v) < 0.$$

If  $a \in (c, 1)$  and  $x$  is a point on  $M$  such that  $(u - av)(x) = \min_X(u - av)$ , then
$$\begin{aligned} u(x) - v(x) &= (u - av)(x) + (a - 1)v(x) \\ &\leq (u - av)(x_0) + (a - 1)v(x) \\ &\leq (u - v)(x_0) + (a - 1)(v(x) - v(x_0)) \\ &\leq (u - v)(x_0) + (1 - a)\text{Osc}_M(v) \end{aligned}$$

Since  $(u - v)(x_0) < 0$ , we can fix  $a \in (c, 1)$ , sufficiently close to 1, such that

$$(u - v)(x_0) + (1 - a)\text{Osc}_M(v) < 0$$

Then, from the above, we can find  $x_a \in \{u < v\}$  such that  $(u - av)(x_a) = \min_M(u - av)$ . The classical maximum principle ensures that at  $x_a$  we have  $dd^c(u - av) \geq 0$ , implying

$$\alpha_u \geq a\alpha_v + (1 - a)\beta$$

because  $(\Delta_\omega(u - av))\omega - dd^c(u - av) \geq 0$ . Since these forms are positive, we obtain

$$(\alpha_u)^n \geq a^n c^{-n} (\alpha_u)^n + (1 - a)^n \beta^n$$

a contradiction, because  $ac^{-1} \geq 1$ , and  $(1 - a)\beta^n > 0$  at  $x_a$ .  $\square$

The following consequences of the arithmetic-geometric mean inequality will be useful:

**Proposition 2.3.** *Given a smooth  $u \in \text{PSH}_{n-1}(M, \beta, \omega)$  such that it solves the complex  $(n-1)$ Monge-Ampère equation  $\alpha_u^n = bg\omega^n$  for a function  $0 < g \in C^\infty(M)$  and some constant  $b > 0$ . Then  $u \in \text{SH}_1(M, \beta, \omega)$  and the following inequality holds:*

$$(nb^{\frac{1}{n}})g^{\frac{1}{n}}\omega^n \leq (\beta + dd^c u) \wedge \omega^{n-1}$$

*Proof.*

$$\begin{aligned} u \in \text{PSH}_{n-1}(M, \beta, \omega) &\implies \beta + \frac{1}{n-1} \left( (\Delta_\omega u)\omega - dd^c u \right) \geq 0, \\ &\stackrel{(\text{Trace})}{\implies} \beta \wedge \omega^{n-1} + \frac{1}{n-1} \left( \left( n \frac{dd^c u \wedge \omega^{n-1}}{\omega^n} \right) \omega^n - dd^c u \wedge \omega^{n-1} \right) \geq 0, \\ &\implies u \in \text{SH}_1(M, \beta, \omega). \end{aligned}$$

Now applying the AM-GM inequality to the eigenvalues of  $\alpha_u$  we get:

$$\begin{aligned} (\beta + dd^c u) \wedge \omega^{n-1} &\geq n \left[ \left( \beta + \frac{1}{n-1} \left( \left( n \frac{dd^c u \wedge \omega^{n-1}}{\omega^n} \right) \omega^n - dd^c u \right) \right)^n \right]^{\frac{1}{n}}, \\ &\geq (nb^{\frac{1}{n}})g^{\frac{1}{n}}\omega^n. \end{aligned}$$

$\square$

**Proposition 2.4.** *Let  $v \in \text{PSH}(M, \omega) \cap C^2(M)$ . Then  $v \in \text{PSH}_{n-1}(M, \omega)$  and  $\omega_v^n \leq \alpha_v^n$ .*

*Proof.*

$$\begin{aligned} \omega + \frac{1}{n-1} \left( \left( n \frac{dd^c v \wedge \omega^{n-1}}{\omega^n} \right) \omega - dd^c v \right) &= \omega + \frac{1}{n-1} \left( \left( n \frac{(dd^c v + \omega) \wedge \omega^{n-1}}{\omega^n} \right) \omega - n\omega - dd^c v \right), \\ &= \frac{1}{n-1} \left( \left( n \frac{(dd^c v + \omega) \wedge \omega^{n-1}}{\omega^n} \right) \omega - (\omega + dd^c v) \right), \\ &= \frac{1}{n-1} (\text{tr}_\omega(\omega_v) \omega - (\omega_v)) \end{aligned}$$

Since  $v \in \text{PSH}(M, \omega)$  then we conclude that  $v \in \text{PSH}_{n-1}(M, \omega)$  from the last computation. Now, from the description through eigenvalues of  $dd^c v$  we have from  $v \in \text{PSH}(M, \omega)$ :

$$1 + \lambda_i \geq 0 \quad \text{and} \quad \frac{n-1 + \widehat{\lambda}_i}{n-1} = \frac{\sum_{j \neq i} (1 + \lambda_j)}{n-1} \geq 0, \forall i \in \{1, \dots, n\}$$

Applying AM-GM to each term above we have:

$$\frac{\sum_{j \neq i} (1 + \lambda_j)}{n-1} \geq \left[ \left( \prod_{j \neq i} 1 + \lambda_j \right) \right]^{\frac{1}{n-1}}, \forall i \in \{1, \dots, n\}$$

By taking the product of the inequalities for all  $i \in \{1, \dots, n\}$  we get:

$$\prod_{i=1}^n \left( 1 + \frac{\widehat{\lambda}_i}{n-1} \right) \geq \prod_{i=1}^n (1 + \lambda_i)$$

which is exactly what we wanted to prove in terms of eigenvalues.  $\square$

For the rest of this section, our equation, for a unknown  $u \in C^2(M) \cap \text{PSH}_{n-1}(M, \rho_\beta \beta, \omega)$  normalized by  $\sup_M u = 0$ , will be on a compact hermitian manifold  $M$ :

$$MA_{n-1}(\rho_\beta \beta, \omega, u) = MA_{n-1}(u) := \frac{\left( \rho_\beta \beta + \left( \frac{(\Delta_\omega u) \omega - dd^c u}{n-1} \right) \right)^n}{\omega^n} = cf, \quad (2.1)$$

for  $(M, \beta, \omega)$  a compact hermitian manifold with two hermitian metrics,  $1 \leq \rho_\beta$  the normalized Gauduchon factor of  $\beta$  by [Gau77a] and  $f \in C^\infty(M)$ .

Our geometric constants under consideration are:

- $\text{Vol}_\omega(M)$  the volume of  $M$  with respect to a metric  $\omega$ ,
- $1 \leq G_\omega$  is the uniform upper bound on the Gauduchon factor  $\rho_\omega$  of a metric  $\omega$  from [Pan22, Theorem A],
- $1 < C_{\beta, \omega}$  is the uniform constant such that  $C_{\beta, \omega}^{-1} \beta \leq \omega \leq C_{\beta, \omega} \beta$ ,
- $0 < A_\alpha, \alpha$  are the uniform constants from Proposition 1.23,
- $0 < v_{m_0}^-(\omega)$  is the uniform constant defined as, given  $0 < m_0$  uniform constant,

$$v_{m_0}^-(\omega) := \inf \left\{ \int_M (\omega + dd^c u)^n : u \in \text{PSH}(M, \omega) \text{ with } -m_0 \leq u \leq 0 \right\}.$$

Such a constant is positive since  $\omega$  is hermitian, by [GL22, Proposition 3.4].

**2.2.  $L^p$  bound on the normalizing constant.** The following result adapts the ideas from [KN15, Lemma 5.9], [Pan23, Lemma 2.7] and [GL25a, Theorem A] we obtain:

**Proposition 2.5.** *The normalizing constant  $c$  in (2.1) admits the two bounds:*

$$C_{\beta,\omega}^{-1}b \leq c \leq \left( \frac{C_{Lap}}{n \|f^{\frac{1}{n}}\|_{L^1(M,\omega^n)}} \right)^n$$

where  $b = b(f, C_{\beta,\omega}^{-1}\omega, A_\alpha) > 0$  is the normalizing constant of the Monge-Ampère equation  $(C_{\beta,\omega}^{-1}\omega + dd^c u)^n = bf\omega^n$  and  $C_{Lap} := \text{Vol}_\omega(M)G_\omega^{n-1}G_\beta C_{\beta,\omega}$ .

See [Pan23, Section 2 and 3] for a precise description of the Monge-Ampère constants.

*Proof.* *Bounding the constant from above.*

By [Pan22, Theorem A], there exists a uniform constant  $G_\omega > 1$  such that the normalized Gauduchon factor  $\rho_\omega$  with respect to  $(M, \omega)$  is bounded between 1 and  $G_\omega$ . For all smooth  $u \in \text{PSH}_{n-1}(M, \rho_\beta\beta, \omega)$ , we calculate

$$\begin{aligned} \int_M (\rho_\beta\beta + dd^c u) \wedge \omega^{n-1} &\leq \int_M (\rho_\beta\beta + dd^c u) \wedge (\rho_\omega\omega)^{n-1} \\ &\leq \int_M \rho_\omega^{n-1} \rho_\beta\beta \wedge \omega^{n-1} + \int_M u dd^c (\rho_\omega\omega)^{n-1} \leq G_\omega^{n-1} G_\beta C_{\beta,\omega} \text{Vol}_\omega(M) \end{aligned}$$

Proposition 2.3 now yields

$$nc^{\frac{1}{n}} f^{\frac{1}{n}} \omega^n \leq (\rho_\beta\beta + dd^c u) \wedge \omega^{n-1}$$

Integrating we can use the upper bound on the mass of the Hessian operator to conclude:

$$nc^{\frac{1}{n}} \|f^{\frac{1}{n}}\|_{L^1(M,\omega^n)} \leq C_{Lap}, \quad \text{hence} \quad c \leq \left( \frac{C_{Lap}}{n \|f^{\frac{1}{n}}\|_{L^1(M,\omega^n)}} \right)^n.$$

*Bounding the constant from below.*

We use the existence of solution to the complex Monge-Ampère equation on hermitian manifolds, that is [TW10, Corollary 1] or [GL23, Theorem B] for the case with two different metrics. Then there exists a pair  $(v, b) \in \text{PSH}(M, C_{\beta,\omega}^{-1}\omega) \cap C^\infty(M) \times \mathbb{R}_{>0}$  such that:

$$(C_{\beta,\omega}^{-1}\omega + dd^c v)^n = bf\omega^n, \quad \sup_M v = 0$$

Now we use Proposition 2.4 to obtain the comparison:

$$bc^{-1}MA_{n-1}(\beta, \omega, u) \leq bf \leq MA_{n-1}(C_{\beta,\omega}^{-1}\omega, C_{\beta,\omega}^{-1}\omega, v) \leq C_{\beta,\omega}MA_{n-1}(\beta, \omega, v)$$

Then, by the domination principle (Lemma 2.2) for  $v$  and  $u$  we get

$$C_{\beta,\omega}^{-1}bc^{-1}\alpha_u^n \leq \alpha_v^n \Rightarrow C_{\beta,\omega}^{-1}bc^{-1} \leq 1. \Rightarrow c \geq C_{\beta,\omega}^{-1}b.$$

By [GL23, Theorem B] the constant  $b$  is bounded from below uniformly, hence so is  $c$ .  $\square$

For the Setting (SF), by [Pan22, Pan23], the constants in Proposition 2.5 are under control.

**2.3.  $L^\infty$  a priori estimate.** Adapting the technique from [GL25b, Theorem B] we prove:

**Theorem 2.6.** *For  $(M, \beta, \omega)$  a compact hermitian manifold with two hermitian metrics, as described above for (2.1). Then, we have the uniform bounds:*

(A):  $c \leq C_0$ ,

$$\text{for } C_0 = \left( \frac{C_{Lap}}{n \|f^{\frac{1}{n}}\|_{L^1(M,\omega^n)}} \right)^n > 0, \text{ where } C_{Lap} := \text{Vol}_\omega(M)G_\omega^{n-1}G_\beta C_{\beta,\omega}$$

- (B):  $-C \leq u \leq 0$ ,  
 for  $C = C(C_{\beta,\omega}^{-1}, \|f\|_{L^p(M,\omega^n)}, n, p, \|u\|_{L^1(M,\omega^n)}, \frac{C_0}{v_{m_0}^-(C_{\beta,\omega}^{-1}\omega)}) > 0$  and a  $m_0 > 0$  uniformly depending on  $f, C_0, C_{\beta,\omega}$ .
- (C):  $c_0 \leq c$ ,  
 for  $c_0 = \frac{C_{\beta,\omega}^n v_{m_0}^-(C_{\beta,\omega}^{-1}\omega)}{\|f\|_{L^1(M,\omega^n)}} > 0$ .

*Proof.* The upper bound on the constant (A) follows exactly as in Proposition 2.5

*Bounding the solution from below.*

To get the second bound (B) we use the comparison with the complex Monge-Ampère through a psh envelope, that is:

Firstly, to alleviate the notation we substitute  $\rho_\beta$  for  $\rho$  and set  $\tilde{\omega} := C_{\beta,\omega}^{-1}\omega \leq \beta$  by (GC). Now consider  $\psi := P_{\tilde{\omega}}(u)$  the  $\tilde{\omega}$ -psh envelope of  $u$ . The function  $\psi$  is  $C^{1,1}$  and the complex Monge-Ampère operator of the envelope is supported on the contact set, i.e.,  $MA_{\tilde{\omega}}(\psi)$  is concentrated on  $\mathcal{C} := \{\psi = u\}$  by [GL22, Theorem 2.3].

It is known that on the contact set  $\mathcal{C}$  the function  $u$  is  $\tilde{\omega}$ -psh. Then on  $\mathcal{C}$  the operator  $MA_{\tilde{\omega}}(u)$  is the complex Monge-Ampère of a  $\tilde{\omega}$ -psh function.

Since the inequality on Proposition 2.4 is a pointwise comparison we have:

$$MA_{\tilde{\omega}}(u)|_{\mathcal{C}} \leq MA_{n-1}(\tilde{\omega}, \tilde{\omega}, u)|_{\mathcal{C}} \leq C_{\beta,\omega}^n MA_{n-1}(\beta, \omega, u)|_{\mathcal{C}} \leq C_{\beta,\omega}^n MA_{n-1}(\rho\beta, \omega, u)|_{\mathcal{C}}$$

at all points of  $\mathcal{C}$ . Since  $u \in \text{PSH}_{n-1}(M, \rho\beta, \omega)$  we get:

$$MA_{\tilde{\omega}}(\psi) = \mathbf{1}_{\mathcal{C}} MA_{\tilde{\omega}}(u) \leq C_{\beta,\omega}^n \mathbf{1}_{\mathcal{C}} MA_{n-1}(\rho\beta, \omega, u) \leq C_{\beta,\omega}^n MA_{n-1}(\rho\beta, \omega, u).$$

Then, we get:

$$MA_{\tilde{\omega}}(\psi) \leq C_{\beta,\omega}^n MA_{n-1}(\rho\beta, \omega, u) = C_{\beta,\omega}^n c f \leq C_{\beta,\omega}^n C_0 f,$$

where the last inequality follows from the bound (A) in the previous step. For a uniformly controlled constant  $m_0 > 0$  we have  $Osc_M(\psi) \leq m_0$ , by [GL23, Theorem A]. Since we have  $\psi \leq u \leq 0$  and the oscillation is under control we have:

$$\int_M MA_{\tilde{\omega}}(\psi) \tilde{\omega}^n \geq v_{m_0}^-(\tilde{\omega})$$

which is under control, since  $m_0$  is under control by [GL22, Proposition 3.4]. We compute:

$$\begin{aligned} 0 &\leq (-\sup_M(\psi))^\delta v_{m_0}^-(\tilde{\omega}) \leq \int_M (-\psi)^\delta MA_{\tilde{\omega}}(\psi) \tilde{\omega}^n, \\ &= \int_{\mathcal{C}} (-\psi)^\delta MA_{\tilde{\omega}}(\psi) \tilde{\omega}^n = \int_M (-u)^\delta MA_{\tilde{\omega}}(\psi) \tilde{\omega}^n, \\ &\leq C_{\beta,\omega}^n C_0 \int_M (-u)^\delta f \tilde{\omega}^n \leq C_0 \|f\|_{L^p(M,\omega^n)} \|u\|_{L^1(M,\omega^n)}^{1/q}. \end{aligned}$$

where the last inequality follows from the Hölder inequality for  $\delta = \frac{1}{q}$ . The term  $\|u\|_{L^1(M,\omega^n)}$  is under control by the compactness in  $L^1$  of  $\text{SH}_1(M, \beta, \omega)$  see [GP24, Lemma 8].

This implies that  $\psi$  is uniformly bounded. Since, by definition  $\psi \leq u$  we conclude:

$$-u \leq m_0 + \left( \frac{C_0 \|f\|_{L^p(M,\omega^n)} \|u\|_{L^1(M,\omega^n)}^{1/q}}{v_{m_0}^-(\tilde{\omega})} \right)^q.$$

*Bounding the constant from below.*

To prove the last bound (C) one can just compare the Monge-Ampère masses:

$$MA_{\tilde{\omega}}(\psi) \leq cf \implies v_{m_0}^-(\tilde{\omega}) \leq c \int_M f \tilde{\omega}^n \implies \frac{v_{m_0}^-(\tilde{\omega})}{C_{\beta,\omega}^{-n} \|f\|_{L^1(M,\omega^n)}} \leq c$$

by integrating with respect to  $\tilde{\omega}$  and since  $\psi$  is uniformly bounded.  $\square$

**2.4. Proof of Theorem A.** In Theorem A, a  $L^\infty$  estimate for the solution  $u_t$  of the family of complex  $(n-1)$ -Monge-Ampère equations, the following constant are under control:

- (1)  $\|f_t^{\frac{1}{n}}\|_{L^1(X_t,\omega_t^n)}, \|f_t\|_{L^p(X_t,\omega_t^n)}$  are under control by assumption of the density bounds (DB).
- (2)  $G_\omega, G_\beta$  are under control since  $\rho_t$  satisfy the Gauduchon bound (GB).
- (3)  $\text{Vol}_{\omega_t}(X_t)$  is under control by the volume bound (VB).
- (4)  $v_{m_0}^-(\omega)$  is uniform, for a  $m_0$  uniform, by [GL22, Proposition 3.4].

Hence, Theorem 2.6 completes the proof of Theorem A.

These bounds above are satisfied in Setting (SF) when  $X_0$  has log-terminal singularities:

- (1) By [DNGG23, Lemma 4.4, equation (4.7)] the adapted volume satisfies (DB).
- (2) By [Pan22, Theorem A], the Gauduchon factors of any smoothable variety satisfy (GB).
- (3) By [Pan22, Section 1], the volume of any smoothable variety satisfies (VB).

### 3. $C^2$ ESTIMATE ON HOLOMORPHIC FAMILIES

From this point onward, we work in a holomorphic family  $\pi : (\mathcal{M}, \beta, \omega) \rightarrow \mathbb{D}$  belongin to Setting (SF), without singular fibers; that is, the central fiber  $M_0 = \pi^{-1}(0)$  is a complex manifold. We further restrict ourself by fixing  $\omega$  to be a Kähler metric, as this is the appropriate setting of the geometric consequence of Theorem B.

In this section, we prove a  $C^2$ -estimate that depends on the norm of the gradient as in [HMW10], who's original strategy traces back to [CW01]. Our proof is a more precise formulation of [TW17, Theorem 4.1] that renders the estimate uniform in families. To help the reader, we align our notation with that of [TW17] and reference their work for the computations that we omit here.

**3.1. Uniform constants.** Firstly we highlight some of the uniform constants that the estimate will depend on fixing their notation. Recall that for this setting  $M$  is a Kähler manifold,  $\omega$  a Kähler metric, however  $\beta$  is hermitian.

**Metric comparison constant** Let  $C_{\beta,\omega} > 1$  be a constant, as in (GC), such that:

$$C_{\beta,\omega}^{-1} \beta \leq \omega \leq C_{\beta,\omega} \beta$$

**Bisectional curvature bound:** Let  $B > 0$  be a constant such that:

$$-B |\xi_1|^2 |\xi_2|^2 \leq R_{i,\bar{j},k,\bar{l}} \xi_1^i \xi_2^{\bar{j}} \xi_2^k \xi_1^{\bar{l}} \leq B |\xi_1|^2 |\xi_2|^2, \quad \xi_1, \xi_2 \in \mathbb{C}^n.$$

**$L^\infty$ -estimate:** Let  $C_{L^\infty} > 0$  be the uniform constant from Theorem 2.6; that is:

$$c + c^{-1} + \sup_M |u| \leq C_{L^\infty}.$$

**Notation:** To align with the notation of [TW17, Section 4] we set the following parallel between the hermitian forms and their Riemannian metrics:

$$\alpha_u = \alpha \rightsquigarrow \tilde{g}, \quad \omega \rightsquigarrow g, \quad \beta \rightsquigarrow h. \quad \text{Hence } \tilde{g}_{i\bar{j}} = h_{i\bar{j}} + \frac{1}{n-1} ((\Delta u)g_{i\bar{j}} - u_{i\bar{j}}).$$

and we are solving the  $(n-1)$ -Monge-Ampère equation:

$$\left( \beta + \frac{(\Delta_\omega u)\omega - dd^c u}{n-1} \right)^n = f\omega^n, \tag{3.1}$$

for  $\sup_M u = 0$  and  $0 < f \in C^\infty(M)$ .<sup>4</sup>

**3.2. Uniform  $C^2$  estimate on manifolds.** We have the following  $C^2$ -estimate:

**Theorem 3.1.** *Suppose  $(M, \beta, \omega)$  is a compact Kähler manifold, where  $\omega$  is a Kähler metric and  $\beta$  is a hermitian metric. Then, there exists a uniform constant  $C > 0$  such that:*

$$\|dd^c u\|_{L^\infty} \leq C(\sup_M |\nabla u|_g^2 + 1)$$

for  $C = C(B, C_{L^\infty}, \|h\|_{C^2}, \|\log(f)\|_{C^2}, \|f\|_{L^\infty}, n, C_{\beta, \omega})$

*Proof.* As in [TW17, Equation (4.2)] we use the tensor  $\eta$ :

$$\eta_{i\bar{j}} = u_{i\bar{j}} - (n-1)h_{i\bar{j}} + (tr_g h)g_{i\bar{j}} = (tr_g \tilde{g})g_{i\bar{j}} - (n-1)\tilde{g}_{i\bar{j}}; \quad \tilde{g}_{i\bar{j}} = \frac{1}{n-1}((tr_g \tilde{g})g_{i\bar{j}} - \eta_{i\bar{j}}) \quad (3.2)$$

We will apply the maximum principle to<sup>5</sup>:

$$H(x, \xi) = \log(\eta_{i\bar{j}}\xi^i\bar{\xi}^j) + \psi_1(|\nabla u|_g^2) + \psi_2(u),$$

for  $x \in M, \xi \in T_x^{1,0}M$  is a  $g$ -unit vector and functions  $\psi_1, \psi_2$  defined as

$$\psi_1(s) = -\frac{1}{2} \log\left(1 - \frac{s}{2K}\right), \quad \text{for } 0 \leq s \leq K-1, \quad \text{for } K = \sup_M |\nabla u|_g^2 + 1. \quad (3.3)$$

$$\psi_2(t) = -A \log\left(1 + \frac{t}{2L}\right), \quad \text{for } -L+1 \leq t \leq 0, \quad \text{for } L = C_{L^\infty} + 1, A = 2L(C_0 + 1). \quad (3.4)$$

for a uniform  $C_0 > 0$  to be chosen later.

We treat  $H$  on a subcompact  $W$  of the  $g$ -unit tangent bundle of  $M$  where  $\eta_{i\bar{j}}\xi^i\bar{\xi}^j \geq 0$ , with  $H = -\infty$  when  $\eta_{i\bar{j}}\xi^i\bar{\xi}^j = 0$ . Hence its maximum is achieved in the interior point  $(x_0, \xi_0) \in \overset{\circ}{W}$ . Pick holomorphic coordinates at  $x_0$  that are  $g$ -normal, as in [TW17, page 323], such that:

$$g_{i\bar{j}} = \delta_{ij}, \quad \eta_{i\bar{j}} = \eta_i \delta_{ij}, \quad \eta_1 \geq \eta_2 \geq \dots \geq \eta_n. \quad (3.5)$$

$$\tilde{g}_{i\bar{j}} = \lambda_i \delta_{ij}, \quad \eta_i = \sum_{k=1}^n \lambda_k - (n-1)\lambda_i, \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n. \quad (3.6)$$

The quantities  $\lambda_n, \eta_1, tr_\omega \alpha$  are uniformly equivalent:

$$\frac{1}{n} tr_g \tilde{g} \leq \lambda_n \leq \eta_1 \leq (n-1)\lambda_n \leq (n-1) tr_g \tilde{g}. \quad (3.7)$$

Since  $\eta_1$  is the largest eigenvalue of  $(\eta_{i\bar{j}})$  the maximum of  $H$  is attained at the vector  $\xi_0 = \frac{\partial}{\partial z^1}$  which extends locally to  $\xi_0 = g_{1\bar{1}}^{-\frac{1}{2}} \frac{\partial}{\partial z^1}$ . Finally, we set:

$$Q(x) = H(x, \xi_0) = \log(g_{1\bar{1}}^{-1} \eta_1) + \psi_1(|\nabla u|_g^2) + \psi_2(u). \quad (3.8)$$

The maximum of  $Q$  is attained at  $x_0$ . We will prove that  $\eta_1(x_0) \leq CK$ , finishing the proof.

Without loss of generality, we may assume  $\eta_1 > 1$  and  $u_{1\bar{1}} > 0$  at  $x_0$ . We define the tensor  $\Theta^{i\bar{j}}$ , and its associated linear operator, as in [TW17, page 324]:

$$\Theta^{i\bar{j}} = \frac{1}{n-1} ((tr_g \tilde{g})g^{i\bar{j}} - \tilde{g}^{i\bar{j}}) > 0, \quad L(v) = \Theta^{i\bar{j}} v_{i\bar{j}} = \frac{1}{n-1} ((tr_g \tilde{g})\Delta v - \tilde{\Delta} v). \quad (3.9)$$

By the choice of  $g$ -normal coordinates, we get  $\nabla \xi_0(x_0) = 0$  for the covariant derivative, yielding the following identities at  $x_0$ , analogous to [TW17, (4.7) and (4.9)]:

$$0 = Q_i = \frac{\eta_{1i}}{\eta_1} + \psi_1' \cdot \left( \sum_p u_p u_{\bar{p}i} + \sum_p u_{pi} u_{\bar{p}} \right) + \psi_2' \cdot u_i. \quad (3.10)$$

<sup>4</sup>The normalizing constant is absorbed into the density  $f$  since we already have a  $L^\infty$ -estimate.

<sup>5</sup>Here our notation slightly differs from [TW17].

Since  $\Theta^{i\bar{j}}$  is diagonal at  $x_0$ , and writing  $(\xi^i)$  for the coordinates of  $\xi_0$ , we obtain:

$$\begin{aligned} L(Q) &= \sum_i \frac{\Theta^{i\bar{i}} \eta_{1i\bar{i}}}{\eta_1} - \sum_i \frac{\Theta^{i\bar{i}} |\eta_{1i}|^2}{(\eta_1)^2} + \sum_i \Theta^{i\bar{i}} (\xi_{i\bar{i}}^1 + \bar{\xi}_{i\bar{i}}^1) + \psi'_2 \sum_i \Theta^{i\bar{i}} u_{i\bar{i}} \\ &\quad + \psi''_2 \sum_i \Theta^{i\bar{i}} |u_i|^2 + \psi''_1 \sum_i \Theta^{i\bar{i}} \left| \sum_p u_p u_{p\bar{i}} + \sum_p u_{p\bar{i}} u_{p\bar{p}} \right|^2 \\ &\quad + \psi'_1 \sum_{i,p} \Theta^{i\bar{i}} |u_{p\bar{i}}|^2 + \psi'_2 \sum_{i,p} \Theta^{i\bar{i}} |u_{pi}|^2 + \psi'_1 \sum_{i,p} \Theta^{i\bar{i}} (u_{p\bar{i}\bar{i}} u_{p\bar{p}} + u_{p\bar{i}\bar{p}} u_p), \end{aligned} \quad (3.11)$$

After tricky direct computations, differentiating covariantly the  $(n-1)$  Monge-Ampère equation and commuting derivaties as in [HMW10, page 552] we get, since  $L(Q) \leq 0$ , at  $x_0$ :

$$\begin{aligned} 0 &\geq \frac{\sum_{i,j} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} (g_{j\bar{i}} \sum_a u_{a\bar{a}\bar{i}} - u_{j\bar{i}\bar{i}} + \hat{h}_{j\bar{i}\bar{i}}) (g_{i\bar{j}} \sum_b u_{b\bar{b}\bar{i}} - u_{i\bar{j}\bar{i}} + \hat{h}_{i\bar{j}\bar{i}})}{(n-1)^2 \eta_1} - \sum_i \frac{\Theta^{i\bar{i}} |\eta_{1i}|^2}{(\eta_1)^2} \\ &\quad + \frac{1}{\eta_1} \left( F_{1\bar{1}\bar{1}} - \sum_i \Theta^{i\bar{i}} \left( \sum_a u_{a\bar{i}} R_{i\bar{a}\bar{1}\bar{1}} - \sum_a u_{a\bar{1}} R_{i\bar{a}\bar{1}\bar{i}} \right) \right. \\ &\quad \left. - \sum_i \tilde{g}^{i\bar{i}} h_{i\bar{i}\bar{1}\bar{1}} - \sum_i \Theta^{i\bar{i}} \left( \hat{h}_{1\bar{1}\bar{i}\bar{i}} - (tr_g h)_{i\bar{i}} \right) \right) \\ &\quad + \psi'_2 \sum_i \Theta^{i\bar{i}} u_{i\bar{i}} + \psi''_2 \sum_i \Theta^{i\bar{i}} |u_i|^2 + \psi''_1 \sum_i \Theta^{i\bar{i}} \left| \sum_p u_p u_{p\bar{i}} + \sum_p u_{p\bar{i}} u_{p\bar{p}} \right|^2 \\ &\quad + \psi'_1 \sum_{i,p} \Theta^{i\bar{i}} |u_{p\bar{i}}|^2 + \psi'_1 \sum_{i,p} \Theta^{i\bar{i}} |u_{pi}|^2 + \psi'_1 \sum_{i,p} \Theta^{i\bar{i}} (u_{p\bar{i}\bar{i}} u_{p\bar{p}} + u_{p\bar{i}\bar{p}} u_p), \end{aligned} \quad (3.12)$$

which corresponds to [TW17, (4.16)], and all the intermediate steps are the same, for  $e^F = f$  and  $\hat{h}_{i\bar{j}} = (n-1)h_{i\bar{j}}$ . Since all the intermediate computations coincide with those in [TW17] we omit them here and focus on describing the constants and inequalities more precisely:

**In [TW17, Equation (4.17)]:** at  $x_0$

$$\sum \Theta^{i\bar{i}} = tr_{\tilde{g}} g, \quad \text{and} \quad |u_{i\bar{j}}| = |\eta_{i\bar{j}} + (n-1)h_{i\bar{j}} - (tr_g h)g_{i\bar{j}}| \leq C_1 \eta_1, \quad (3.13)$$

for  $C_1 = 1 + (2n-1)\|h\|_{L^\infty}$ , by (3.2) and since  $\eta_1 > 1$  is the highest eigenvalue of  $(\eta_{i\bar{j}})$  at  $x_0$ .

We now provide lower bounds for the second and third lines of (3.12)

**In [TW17, Equations (4.18) and (4.19)]:**

$$\begin{aligned} &\frac{1}{\eta_1} \left( F_{1\bar{1}\bar{1}} - \sum_i \Theta^{i\bar{i}} \left( \sum_a u_{a\bar{i}} R_{i\bar{a}\bar{1}\bar{1}} - \sum_a u_{a\bar{1}} R_{i\bar{a}\bar{1}\bar{i}} \right) \right. \\ &\quad \left. - \sum_i \tilde{g}^{i\bar{i}} h_{i\bar{i}\bar{1}\bar{1}} - \sum_i \Theta^{i\bar{i}} \left( \hat{h}_{1\bar{1}\bar{i}\bar{i}} - (tr_g h)_{i\bar{i}} \right) \right) \geq -C_2 tr_{\tilde{g}} g - C_3, \end{aligned} \quad (3.14)$$

for  $C_2 = 2n(\|h\|_{C^2} + C_1 B)$  and  $C_3 = \|F\|_{C^2}$ . Since  $\eta_1 > 1$  and using equations (3.13) with the bound on the bisectional curvature  $B$ . And by taking the  $\tilde{g}$ -trace of  $\tilde{g}$  we get:

$$\sum_i \Theta^{i\bar{i}} u_{i\bar{i}} = \frac{1}{n-1} ((tr_{\tilde{g}} g) \Delta u - \tilde{\Delta} u) = n - tr_{\tilde{g}} h, \quad (3.15)$$

Now we will bound the last term written in (3.12):

**In [TW17, Equations (4.20) and (4.23)]:**

Again by using the covariant differentiation of the PDE and commuting derivatives:

$$\begin{aligned}
& \psi'_1 \sum_{i,p} \Theta^{i\bar{i}} (u_{p i \bar{i}} u_{\bar{p}} + u_{\bar{p} i \bar{i}} u_p) \\
&= \psi'_1 \sum_p (F_p u_{\bar{p}} + F_{\bar{p}} u_p) - \psi'_1 \sum_{i,p} \tilde{g}^{i\bar{i}} (h_{i\bar{i}p} u_{\bar{p}} + h_{i\bar{i}\bar{p}} u_p) \\
&\quad + \psi'_1 \sum_{i,p,q} \Theta^{i\bar{i}} u_q u_{\bar{p}} R_{p\bar{q}i\bar{i}} \\
&\geq - \|F\|_{C^1} (1 + |\nabla u|_g^2) |\psi'_1| - (\|h\|_{C^1} + n^2 B) \text{tr}_{\tilde{g}} g (|\nabla u|_g^2 + 1) |\psi'_1| \\
&\geq - C_4 \text{tr}_{\tilde{g}} g - C_5.
\end{aligned} \tag{3.16}$$

for  $C_4 = \frac{\|h\|_{C^1} + n^2 B}{2}$  and  $C_5 = \frac{\|F\|_{C^1}}{2}$ . By using the Cauchy-Buniakovski-Schwarz inequality, the bound on the bisectional curvature  $B$ ,  $\frac{1}{2K} \geq \psi'_1 \geq \frac{1}{4K}$  and (3.13).

Combining (3.12) with (3.14), (3.15) and (3.16) yields:

$$\begin{aligned}
0 &\geq \frac{\sum_{i,j} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} (g_{j\bar{i}} \sum_a u_{a\bar{a}\bar{i}} - u_{j\bar{i}} + \hat{h}_{j\bar{i}}) (g_{i\bar{j}} \sum_b u_{b\bar{b}i} - u_{i\bar{j}} + \hat{h}_{i\bar{j}})}{(n-1)^2 \eta_1} - \sum_i \frac{\Theta^{i\bar{i}} |\eta_{1i}|^2}{(\eta_1)^2} \\
&\quad + \psi'_2 \sum_i \Theta^{i\bar{i}} |u_i|^2 + \psi''_1 \sum_i \Theta^{i\bar{i}} \left| \sum_p u_p u_{\bar{p}i} + \sum_p u_{pi} u_{\bar{p}} \right|^2 + \psi'_1 \sum_{i,p} \Theta^{i\bar{i}} |u_{p\bar{i}}|^2 \\
&\quad + \psi'_1 \sum_{i,p} \Theta^{i\bar{i}} |u_{pi}|^2 - \text{tr}_{\tilde{g}} g (C_{\beta,\omega} \psi'_2 + C_2 + C_4) - (n\psi'_2 + C_3 + C_5),
\end{aligned} \tag{3.17}$$

since  $-\frac{A}{L} \leq \psi'_2 \leq -\frac{A}{2L} = -(C_0 + 1)$ . By choosing  $C_0 = \frac{C_2 + C_4}{C_{\beta,\omega}} + 2 \frac{n \|h\|_{C^1}^2}{(n-1) C_{\beta,\omega}}$ , the only term left to control is  $\sum_i \frac{\Theta^{i\bar{i}} |\eta_{1i}|^2}{(\eta_1)^2}$ , as all other terms are positive. As in [TW17, page 328] we define:

$$\delta = \frac{1}{1 + 2A} = \frac{1}{1 + 4L(C_0 + 1)},$$

which is uniform because so are  $L$  and  $C_0$ . Now we have two cases, as in [HMW10, TW17].

**Case 1** (Assume  $\lambda_2 \leq (1 - \delta)\lambda_n$ ). We use (3.10) and (3.13) to obtain

$$\begin{aligned}
-\sum_i \frac{\Theta^{i\bar{i}} |\eta_{1i}|^2}{(\eta_1)^2} &= -\sum_i \Theta^{i\bar{i}} \left| \psi'_1 \left( \sum_p u_p u_{\bar{p}i} + \sum_p u_{pi} u_{\bar{p}} \right) + \psi'_2 u_i \right|^2 \\
&\quad - 2(\psi'_1)^2 \sum_i \Theta^{i\bar{i}} \left| \sum_p u_p u_{\bar{p}i} + \sum_p u_{pi} u_{\bar{p}} \right|^2 - 8(C_0 + 1)^2 K \text{tr}_{\tilde{g}} g,
\end{aligned} \tag{3.18}$$

since  $|\psi'_2| \leq \frac{A}{L} = 2(C_0 + 1)$ . Then, using  $\psi''_1 = 2(\psi'_1)^2$ ,  $\psi''_2 > 0$ ,  $\psi'_1, -\psi'_2 - C_0 \geq 1$ ,  $\psi'_1 \geq \frac{1}{4K}$  and erasing positive unhelpful terms we obtain from (3.17)

$$0 \geq \frac{1}{4K} \Theta^{n\bar{n}} u_{n\bar{n}} - 8(C_0 + 1)^2 K \text{tr}_{\tilde{g}} g - C_6, \tag{3.19}$$

for  $C_6 = C_3 + C_5$ . Since  $\Theta^{n\bar{n}} \geq \frac{1}{n} \sum_i \Theta^{i\bar{i}} \geq \frac{1}{n} \text{tr}_{\tilde{g}} g$ :

$$\frac{1}{n} (\text{tr}_{\tilde{g}} g) u_{n\bar{n}}^2 \leq 32(C_0 + 1)^2 K^2 \text{tr}_{\tilde{g}} g + C_6 K. \tag{3.20}$$

By the assumption  $\lambda_2 \leq (1 - \delta)\lambda_n$  and (3.2)

$$u_{n\bar{n}} \leq \lambda_1 - \delta \lambda_n + (2n - 1) \|h\|_{L^\infty} \leq -\frac{\delta}{2} \lambda_n$$

since  $\delta < 1$ , and without loss of generality,  $\lambda_1 < 1$  and  $\frac{\lambda_n}{2} \geq \frac{(2n-1)\|h\|_{L^\infty}+1}{\delta}$ . Thus  $u_{n\bar{n}}^2 \geq \frac{\delta^2}{4}\lambda_n$ , and combining with (3.20), we conclude

$$\lambda_n \leq C_7 K$$

for  $C_7 = \frac{8nC_6}{\delta^2}$ , using  $1 \leq \text{tr}_{\bar{g}} g \cdot \lambda_n$  and, without loss of generality,  $\frac{\lambda_n}{2} \geq \frac{32(C_0+1)^2 K^2 (4n)}{\delta^2}$ . By (3.7), this implies the desired estimate  $\eta_1 \leq (n-1)C_7 K$ .

**Case 2** (Assume  $\lambda_2 \geq (1-\delta)\lambda_n$ ). Similar to Case 1, we have for the summand  $i = 1$

$$-\frac{\Theta^{1\bar{1}}|\eta_{1\bar{1}}|^2}{(\eta_1)^2} \geq -2(\psi'_1)^2 \Theta^{1\bar{1}} \left| \sum_p u_p u_{\bar{p}i} + \sum_p u_{pi} u_{\bar{p}} \right|^2 - 8(C_0+1)^2 n K \Theta^{1\bar{1}},$$

combining it with  $2(\psi'_1)^2 = \psi''_1$ ,  $\psi'_1 \geq \frac{1}{4K}$ ,  $-C_{\beta,\omega}\psi'_2 - C_2 - C_4 \geq 2\frac{n\|h\|_{C^1}^2}{(n-1)}$  and (3.17)

$$\begin{aligned} 0 &\geq \frac{\sum_{i,j} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} (g_{j\bar{i}} \sum_a u_{a\bar{a}\bar{i}} - u_{j\bar{i}\bar{i}} + \hat{h}_{j\bar{i}\bar{i}}) (g_{i\bar{j}} \sum_b u_{b\bar{b}\bar{i}} - u_{i\bar{j}\bar{i}} + \hat{h}_{i\bar{j}\bar{i}})}{(n-1)^2 \eta_1} - \sum_{i=2}^n \frac{\Theta^{i\bar{i}} |\eta_{1i}|^2}{(\eta_1)^2} \\ &\quad + \psi''_2 \sum_i \Theta^{i\bar{i}} |u_i|^2 + \psi''_1 \sum_{i=1}^n \Theta^{i\bar{i}} \left| \sum_p u_p u_{\bar{p}i} + \sum_p u_{pi} u_{\bar{p}} \right|^2 \\ &\quad + \frac{1}{4K} \sum_{i,p} \Theta^{i\bar{i}} |u_{pi}|^2 + C_8 \text{tr}_{\bar{g}} g - 8(C_0+1)^2 n K \Theta^{1\bar{1}} - C_6, \end{aligned} \quad (3.21)$$

for  $C_8 = 2\frac{n}{n-1}\|h\|_{C^1}^2$ . Without loss of generality, we assume  $\frac{1}{4K}\Theta^{1\bar{1}}u_{1\bar{1}}^2 \geq 8(C_0+1)^2 n K \Theta^{1\bar{1}}$  then with Lemma 3.2 below, which is a quantitative version of [TW17, Lemma 4.3], we conclude  $0 \geq \frac{C_8}{2} \text{tr}_{\bar{g}} g - C_6$  yielding the desired estimate

$$\eta_1 \leq (n-1)\|f\|_{L^\infty} \left( \frac{2C_6}{C_8} \right)^{n-1}.$$

□

**Lemma 3.2.** *We have, at  $x_0$ :*

$$\begin{aligned} E := & \frac{\sum_{i,j} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} (g_{j\bar{i}} \sum_a u_{a\bar{a}\bar{i}} - u_{j\bar{i}\bar{i}} + \hat{h}_{j\bar{i}\bar{i}}) (g_{i\bar{j}} \sum_b u_{b\bar{b}\bar{i}} - u_{i\bar{j}\bar{i}} + \hat{h}_{i\bar{j}\bar{i}})}{(n-1)^2 \eta_1} - \sum_{i=2}^n \frac{\Theta^{i\bar{i}} |\eta_{1i}|^2}{(\eta_1)^2} \\ & + \psi''_2 \sum_i \Theta^{i\bar{i}} |u_i|^2 + \psi''_1 \sum_{i=1}^n \Theta^{i\bar{i}} \left| \sum_p u_p u_{\bar{p}i} + \sum_p u_{pi} u_{\bar{p}} \right|^2 \geq -\frac{C_8}{2} \text{tr}_{\bar{g}} g \end{aligned}$$

for  $C_8 = 2\frac{n}{n-1}\|h\|_{C^1}^2$ .

*Proof.* Using (3.10),  $\psi''_1 = 2(\psi_1)^2$  and [HMW10, Proposition 2.3], which states that for all  $a, b \in \mathbb{C}^n$  and  $0 < \delta' < 1$  we have  $|a+b|^2 \geq \delta'|a|^2 - \frac{\delta'}{1-\delta'}|b|^2$ , we obtain

$$\psi''_1 \sum_{i=2}^n \Theta^{i\bar{i}} \left| \sum_p u_p u_{\bar{p}i} + \sum_p u_{pi} u_{\bar{p}} \right|^2 \geq 2\delta \sum_{i=2}^n \Theta^{i\bar{i}} \frac{|\eta_{1i}|^2}{(\eta_1)^2} - \frac{2\delta(\psi'_2)^2}{1-\delta} \sum_{i=2}^n \Theta^{i\bar{i}} |u_i|^2 \quad (3.22)$$

Combining it with the fact that  $\psi''_2 \geq \frac{2\varepsilon}{1-\varepsilon}(\psi'_2)^2$  for all  $\varepsilon \leq \frac{1}{2A+1}$ , recall  $\delta = \frac{1}{2A+1}$ , we obtain:

$$E \geq \frac{\sum_{i,j} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} (g_{j\bar{i}} \sum_a u_{a\bar{a}\bar{i}} - u_{j\bar{i}\bar{i}} + \hat{h}_{j\bar{i}\bar{i}}) (g_{i\bar{j}} \sum_b u_{b\bar{b}\bar{i}} - u_{i\bar{j}\bar{i}} + \hat{h}_{i\bar{j}\bar{i}})}{(n-1)^2 \eta_1} - (1-2\delta) \sum_{i=2}^n \frac{\Theta^{i\bar{i}} |\eta_{1i}|^2}{(\eta_1)^2} \quad (3.23)$$

Commuting derivatives, erasing unhelpful positive terms, using (3.2) at  $x_0$ ,  $g_{1\bar{i}} = 0$  for  $i > 1$  and [HMW10, Proposition 2.3] again, we obtain:

$$\begin{aligned} & \frac{\sum_{i,j} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} (g_{j\bar{i}} \sum_a u_{a\bar{a}\bar{i}} - u_{j\bar{i}\bar{i}} + \hat{h}_{j\bar{i}\bar{i}}) (g_{i\bar{j}} \sum_b u_{b\bar{b}\bar{i}} - u_{i\bar{j}\bar{i}} + \hat{h}_{i\bar{j}\bar{i}})}{(n-1)^2 \eta_1} \\ & \geq (1 - \frac{\delta}{2}) \sum_{i=2}^n \frac{\tilde{g}^{i\bar{i}} \tilde{g}^{1\bar{1}} |\eta_{1\bar{i}}|^2}{(n-1)^2 \eta_1} - \frac{1 - \frac{\delta}{2}}{\frac{\delta}{2}} \sum_{i=2}^n \frac{\tilde{g}^{i\bar{i}} \tilde{g}^{1\bar{1}} |(tr_g h) g_{1\bar{i}}|_i^2}{(n-1)^2 \eta_1}. \end{aligned} \quad (3.24)$$

In Case 2, we have  $\tilde{g}^{i\bar{i}} \leq \frac{1}{(1-\delta)\lambda_n}$  for  $i > 1$ . Without loss of generality, we can assume  $(1-\delta)\lambda_n \geq \frac{1-\frac{\delta}{2}}{\frac{\delta}{2}}$ . Then

$$\frac{1 - \frac{\delta}{2}}{\frac{\delta}{2}} \sum_{i=2}^n \frac{\tilde{g}^{i\bar{i}} \tilde{g}^{1\bar{1}} |(tr_g h) g_{1\bar{i}}|_i^2}{(n-1)^2 \eta_1} \leq \frac{C_8}{2} tr_{\tilde{g}} g,$$

for  $C_8 = 2 \frac{n \|h\|_{C^1}^2}{n-1}$ , because the coordinates around  $x_0$  are  $g$ -normal. It remains to prove

$$(1 - \frac{\delta}{2}) \sum_{i=2}^n \frac{\tilde{g}^{i\bar{i}} \tilde{g}^{1\bar{1}} |\eta_{1\bar{i}}|^2}{(n-1)^2 \eta_1} \geq (1 - 2\delta) \sum_{i=2}^n \Theta^{i\bar{i}} \frac{|\eta_{1\bar{i}}|^2}{(\eta_1)^2},$$

which is precisely [TW17, Equation (4.39)]; we refer the reader to it for the proof.  $\square$

**3.3. Proof of the  $C^2$  estimate on holomorphic families.** From Theorem 3.1, the constant in the  $C^2$ -estimate depends on:

- (1)  $B > 0$ , uniform upper and lower bound on the bisectional curvature.
- (2)  $C_{L^\infty}$ , the uniform  $L^\infty$  bound on the solution to (3.1).
- (3) The  $C^2$  norms of  $\log(f)$ , the metrics  $\omega$  and  $\beta$ , and the  $L^\infty$  norm of  $f$ .

Under Setting (SF) without singular fibers, these constants are uniform:

- (1) By compactness of the fibers, the constant  $B$  can be taken uniform on compact subsets of the base manifold  $\mathbb{D}$ , hence it is uniform in the holomorphic family.
- (2) By Theorem 2.6, the constant  $C_{L^\infty}$  is uniform even when the family has singularities.
- (3) By compactness, all these norms are uniform on compact subsets of  $\mathbb{D}$ .

Therefore, the  $C^2$  bound, which depends on  $|\nabla u|_g^2$ , is uniform along the holomorphic family.

## 4. PROOF OF THEOREM B

**4.1.  $C^1$  Estimate on holomorphic families.** Since the  $C^2$ -estimate in the previous section depends on  $\sup_M |\nabla u|_g^2$  we adapt the blow-up argument of [TW17], based on [DK17], to obtain a  $C^1$  estimate for holomorphic families.

**Theorem 4.1.** *Let  $(\mathcal{M}, \beta, \omega)$  be a holomorphic family in the fixed setting. Then there is a constant  $C > 0$  depending only on  $\|f\|_{C^2(\mathcal{M}, \omega)}$  and  $(M, \beta_0, \omega_0)$ , such that*

$$\sup_{\mathbb{D}_{\frac{1}{2}}} |\nabla u|_\omega \leq C.$$

*In particular, the constant  $C$  does not depend on the complex structure.*

*Proof.* Assume by contradiction that  $\|f\|_{C^2(\mathcal{M}, \omega)} \leq C$  uniformly and we have  $u_t, C_t$  such that:

$$\sup_{M_t} |\nabla u_t|_{\omega_t} = C_t$$

with  $C_t \rightarrow +\infty$  as  $t \rightarrow t_0$ , for simplicity set  $t_0 = 0$ . Choose a sequence  $t_j \rightarrow 0$  (denoted simply by  $j$ ) so that  $\sup_{M_j} |\nabla u_j|_{\omega_j} = C_j$ , with  $C_j \rightarrow +\infty$  as  $j \rightarrow +\infty$ .

We have:  $\sup_{M_j} u_j = 0$ ,  $\sup_{M_j} |u_j| \leq C$  (independent of  $j$  from the  $L^\infty$ -estimate) and

$$\alpha_j = \beta_j + \frac{1}{n-1} \left( (\Delta_j u_j) \omega_j - dd^t u_j \right), \quad \alpha_j^n = f_j \omega_j^n.$$

Let  $x_j = \operatorname{arccmax}_{M_j} |\nabla u_j|$ . Up to extracting a subsequence  $x_j \rightarrow x_0 \in M = M_0$ , for  $t_j \rightarrow 0$ .

Since the family is holomorphic then locally one has  $V \subset \mathcal{M}$  open such that  $V_t \subset X_t$ ,  $V_0 \subset X = X_0$ ,  $t \in I \subset B$  and set  $V = B_2(0) \times I$  such that  $\omega(0) = \omega_{\mathbb{C}^n} = \sum_j idz^j \wedge d\bar{z}^j$  with the notation  $\omega = \omega_0$  for the metric on fiber the  $X$  whenever confusion can be avoided.

Assume for  $j$  large enough  $x_{t_j}$  are all in  $B_1(0) \times I'$ ;  $I' \Subset I$ . Define in  $B_{C_j}(0) \times I \subset \mathbb{C}^n \times I$ :

$$\widehat{u}(z, t) = u_j(C_j^{-1}z + x_j; t); \quad \Phi_j : \mathbb{C}^n \rightarrow \mathbb{C}^n, \Phi_j(z) = C_j^{-1}z + x_j$$

Since the family is holomorphically trivial, we will have  $u_j, u$  well defined in small neighbourhoods and denote for simplicity as if all the points are on  $B_1(0) \subset X_0$ . Then we have:

$$\sup_{B_{C_j}(0)} |\widehat{u}_{t,j}| \leq C; \forall t \in I, \forall j \text{ big enough, for } \widehat{u}_{t,j} = \widehat{u}_j(\cdot, t).$$

Fixing  $t \in I$  we have

$$\nabla^{g_t} \widehat{u}_{t,j} = \nabla^{g_t} u_{t,j} C_{t,j}^{-1} Id$$

then  $|\nabla^{g_t} \widehat{u}_{t,j}|(0) = 1$ . Thus  $\sup_{X_j} |\nabla^{g_j} \widehat{u}_j| \leq C$ , independent of  $t, j$ . By the  $C^2$  estimate:

$$\sup_{B_{C_j}(0)} |\partial \bar{\partial} \widehat{u}_j|_{\omega_{\mathbb{C}^n}} \leq C C_j^{-2} \sup_{X_j} |\partial \bar{\partial} u_j|_{\omega} \leq C$$

As in [TW17] by the Sobolev embedding and the elliptic estimates for  $\Delta$  for each given  $K \subset \mathbb{C}^n$  compact,  $0 < \gamma < 1$  and  $p > 1$ , there is a constant  $C$  such that:

$$\|\widehat{u}_j\|_{C^{1,\gamma}(K)} + \|\widehat{u}_j\|_{W^{2,p}(K)} \leq C.$$

Hence, there is a subsequence of  $\widehat{u}_j$  that converges to  $U$  in  $\mathbb{C}^n$  with  $\sup_{\mathbb{C}^n} (|U| + |\nabla U|) \leq C$  and  $\nabla U(0) \neq 0$ , consequently non-constant. Finally, the rest of the proof follows verbatim the one in [TW17] That is, one proves that  $U$  is maximal in  $\mathbb{C}^n$ , hence satisfy all the hypothesis in Theorem 1.24, which implies  $U$  is constant, arriving at a contradiction.  $\square$

**4.2. Higher-Order Estimates on holomorphic families.** As in the classical Monge-Ampère theory, higher-order estimates are purely local. Hence, one can adapt a complex Evans-Krylov type argument as in [TW17, Theorem 6.1] to obtain a  $C^{2,\gamma}$ -estimate, that depend on two derivatives of  $\log(f)$ , and then apply Schauder estimates with a bootstrapping argument to conclude Theorem B. Moreover, the Evans-Krylov-Caffarelli theory developed in [TWWY15, Theorems 1.1 and 1.2] yields estimates that depend only on Hölder bounds of  $\log(f)$ , the metrics, and the  $C^2$  estimate on the solution. Implying explicitly that the estimates remain uniform on the holomorphic family by Sections 3 and 4.1.

This completes the proof of Theorem B.

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